Dynamic Contracting with Flexible Monitoring*

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Abstract

We study a dynamic contracting problem in which the principal can allocate his limited capacity between seeking evidence that confirms or that contradicts the agent's effort, as the basis for reward or punishment. Such flexibility calls for jointly designed monitoring and compensation schemes practically relevant but novel in the literature. When the agent's continuation value is low, the principal seeks only confirmatory evidence, but when the agent's continuation value exceeds a threshold, the principal switches to seeking mainly contradictory evidence. Moreover, the agent's effort can be perpetuated if and only if both synergy and flexibility in monitoring are sufficiently large.

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1 Introduction

In a principal-agent relationship, rather than passively relying on existing performance indicators, the principal ("he") must often actively investigate whether the agent ("she") is taking his desired action.[1][2] This is especially the case when the agent's action has only a long-run impact that is not well reflected by existing indicators. For example, consider a firm that requires its manager's effort in the process of building up the firm's intangible assets or long-run reputation. Since the benefit of such effort is realized only in the long run, whether the manager is exerting effort may not be reflected in current sales, output or stock prices. How should the principal allocate his limited resources in such an investigation? Should he focus more on confirming or refuting the fact that the agent is taking his desired action? What should the associated reward or penalty be? How does his decision depend on their past interaction? These are common issues in real life, such as in human resource management, corporate governance, bureaucratic systems and educational practices.

To tackle these issues, we need a *dynamic* framework that features the *joint* design of monitoring and compensation schemes, which allows the principal to *flexibly* allocate his limited monitoring capacity between seeking confirmatory and contradictory evidence of the agent's effort. However, the existing literature either assumes a single exogenous performance indicator, or focuses on how much monitoring capacity should be devoted to a *given* performance indicator. This paper fills that gap. With our framework, we find that the flexibility of the principal in allocating monitoring capacity, together with the magnitude of the latent benefit, plays an important role in shaping the optimal contract and determining whether the project can be long lasting.

To establish a conceptual framework, consider a continuous-time setup, in which the principal ("he") has a project that requires an agent ("she") to operate. The agent is less patient than the principal and can work or shirk at

[1] We do not intentionally associate the players with particular genders.

[2] The literature on this issue dates back to (Townsend 1979).

each instant. From the perspectives of both the principal and social welfare, it is optimal for the agent to work, but the agent enjoys a private benefit from shirking. To incentivize the agent, at each instant, the principal chooses a combination of "carrot-based search" ("C-search" hereafter) and "stick-based search" ("S-search" hereafter). That is, he can allocate his fixed amount of monitoring capacity to seek two types of evidence, and can determine how much to reward or punish the agent upon its receipt. C-evidence confirms the agent's effort as it emerges only if the agent has worked, while S-evidence refutes the agent's effort as it emerges only if the agent has shirked. The principal can also terminate the project at any time, which is socially inefficient.

In addition to the standard incentive versus interest tradeoff in (DeMarzo and Sannikov 2006) and (Sannikov 2008), the principal faces a tradeoff between C-search and S-search as a means to incentivize the agent. On one hand, C-search generates greater variation than S-search in the agent's continuation value, and are thus less advantageous to the principal, who is effectively risk averse in the relevant range of the agent's continuation value. This is because, given that the agent indeed works, there would be no S-evidence and thus no adjustment to the agent's continuation value is required; while C-evidence does emerge in equilibrium, which necessarily involves a reward upon its receipt ("carrots" hereafter) and the downward adjustment of the agent's continuation value in the absence of C-evidence. On the other hand, for S-search alone to be an effective incentive, a sufficiently high continuation value is required as the agent's stake in the project, whereas the effectiveness of C-search does not depend on the agent's continuation value. Moreover, even if S-search can work alone, a high continuation value for the agent has to be maintained, which involves interest expenditure for the principal, making S-search less advantageous than C-search. This tradeoff between C-search and S-search, together with the incentive versus interest tradeoff, shapes the optimal incentive scheme.

When the agent's continuation value is low, the principal allocates all his monitoring capacity to C-search. Instead of paying the agent immediately upon receiving C-evidence, the principal adds the whole reward to the agent's continuation value, in order to build a buffer against inefficient termination and to make S-search effective in the future. In addition, since the arrival rate of C-evidence is set to its maximum, carrots should be just enough to deter shirking.

When the agent's continuation value has reached a level sufficient for S-search to be effective, but is not enough for S-search alone to deter shirking, the optimal incentive scheme features a "phase change". That is, instead of the carrot-only mode, the principal now relies mainly on S-search, and sets the penalty for observing S-evidence ("sticks" hereafter) to its maximum: confiscation of the whole stake promised to the agent, resulting in termination of the project. C-search are still used to make up for the S-search, but carrots are larger to minimize the reliance on C-search. Carrots decrease as the agent's continuation value grows further.

When the agent's continuation value grows beyond the payout boundary, the conflict of interest between the principal and the agent is so small that there is little need to further incentivize the agent. But the interest accrued from deferred payment is large. Thus, it is optimal for the principal to make payment at once, so as to reduce the agent's continuation value back to the payout boundary.

A novel feature of our model relative to the existing literature concerns the perpetuation of the agent's effort and the convexity of the value function. The flexibility of combining C-search with S-search offers the principal the option of first building up the agent's stake in the game (i.e., her continuation value) with C-search, and then perpetuating the agent's effort mainly with S-search, which avoids inefficient termination. We show that such an option is optimal if and only if the latent benefit from the agent's effort *and* the principal's maximum feasible reliance on S-search are *both* sufficiently large. Moreover, when perpetuation of the agent's effort is optimal, the value function is convex in the vicinity of the (absorbing) payout boundary if public randomization is not allowed. This is due to a new economic force in addition to the standard incentive versus interest tradeoff. That is, the higher the agent's continuation value, not only is it the less likely to reach the (inefficient) termination boundary as in existing models, but it is also more likely to reach the absorbing payout boundary, where the project becomes completely immune to inefficient termination. The latter fact makes the marginal benefit of raising the agent's continuation value increasing instead of decreasing in the continuation value in the vicinity of the payout boundary.

Our model also yields empirically plausible predictions. First, a firm's junior employees are incentivized mainly based on evidence that confirms their contribution to their employer, while senior employees are incentivized mainly based on evidence that refutes their contribution. Second, concerning the compensation scheme, the reward for each piece of evidence that confirms a contribution to the employer varies little among junior employees, but decreases with seniority for senior employees, and features an upward jump when a junior employee becomes senior. The penalty for each piece of evidence that contradicts a contribution to the employer increases with seniority for both junior and senior employees. Third, except for those hired permanently, all employees are more prone to unemployment in the absence of evidence that confirms their contribution, and more so if the employees are less senior. Lastly, employers offer permanent positions if and only if both their flexibility in adjusting monitoring schemes and the potential synergy created by employees are sufficiently large.

1.1 Literature Review

Our work is related mainly to the continuous-time dynamic contracting literature, pioneered by (DeMarzo and Sannikov 2006), (Biais, Mariotti, Plantin, and Rochet 2007) and (Sannikov 2008). Both (DeMarzo and Sannikov 2006) and (Biais, Mariotti, Plantin, and Rochet 2007) study continuous-time variants of the discrete-time dynamic security design model in (DeMarzo and Fishman 2007). (DeMarzo and Sannikov 2006) directly apply the martingale representation technique developed in (Sannikov 2008) in a continuous-time setup, while (Biais, Mariotti, Plantin, and Rochet 2007) is based on the continuous-time limit of a discrete-time model. Early work on dynamic moral hazard models also includes (Biais, Mariotti, Rochet, and Villeneuve 2010). Like our model, (Biais, Mariotti, Rochet, and Villeneuve 2010) use a Poisson process instead of Brownian motions to model discrete losses in continuous time, whose arrival rate depends on the hidden action of the agent. (Myerson 2015) considers a similar problem under a political economics framework where a political leader uses randomized punishment to motivate governors to work. In contrast to the discrete losses in (Biais, Mariotti, Rochet, and Villeneuve 2010), (Sun and Tian 2017) use Poisson processes to model arrivals of discrete revenue. Similarly, (He 2012) considers a risk-averse agent who can save privately and whose hidden effort affects the arrival rate of discrete revenue. In those models, monitoring technologies are exogenous. In other words, the output processes, which are functions of hidden actions and other random factors, are exogenously assumed, and play dual roles as both direct determinants of physical payoff and bases for monitoring and contracting. The essence of our model is to separate these two roles in order to study the interaction between the design of monitoring technology and that of contracts.

Recent work also endogenizes the monitoring scheme in dynamic moral hazard models. On top of the framework of (DeMarzo and Sannikov 2006), (Piskorski and Westerfield 2016) allow the principal to monitor the agent at a cost that increases with his monitoring intensity. Based on a framework similar to that of (Biais, Mariotti, Rochet, and Villeneuve 2010), (Chen, Sun, and Xiao 2017) consider the timing decision of monitoring, where monitoring is modeled as paying a fixed cost for a credible guarantee of the agent taking the desired action. (Varas, Marinovic, and Skrzypacz 2019) consider a problem where monitoring serves as an incentive device and also provides information to the principal. In (Orlov 2018), the principal can change his monitoring intensity. While these papers probe into *how much* monitoring capacity should be devoted to a *given* monitoring technology and its optimal timing, our focus is on the principal's *optimal allocation* of monitoring capacity to *multiple* information sources, as the basis for both his monitoring activities and the design of his incentive scheme.

In a static setup, (Li and Yang 2019) and (Georgiadis and Szentes 2019) also study the impact of the principal's flexibility on the design of his monitoring scheme. Based instead on a dynamic setup, we are able to explore when it is optimal for the principal to perpetuate the agent's effort. In addition, our notion of flexibility is different from that in (Georgiadis and Szentes 2019). The information source in (Georgiadis and Szentes 2019) is a *single* exogenous (conditional on the agent's effort) linear diffusion process, and the flexibility that they consider refers to the freedom of the principal to stop observing that process earlier if existing observations are sufficient to prove the agent's deviation from the desired action. Instead, the notion of flexibility in our paper refers to the freedom of the principal to allocate different levels of monitoring capacity to *various* processes (interpreted as different performance indicators) contingent on the whole history summarized by the agent's continuation value.

Our work is also related to the literature on problems of dynamic attention allocation. (Nikandrova and Pancs 2018) analyze a dynamic problem in which an investor decides how to allocate his limited attention between seeking confirmatory evidence of the profitability of one project and seeking that of another project. Also in a dynamic setting, (Che and Mierendorff 2019) study an individual's decision among immediate action, confirmatory learning (i.e., seeking evidence that would confirm the state he finds relatively more likely) and contradictory learning (i.e., seeking evidence that would confirm the state he finds relatively less likely), before taking actions that affect his state-contingent payoff. Similar to (Che and Mierendorff 2019), (Kuvalekar and Ravi 2019) consider how a principal should incentivize an agent, who is to allocate limited attention between seeking evidence that confirms and evidence refutes the quality of a project. While the monitoring capacity allocation between C-search and S-search in our model is similar to the learning problems in these papers, the problem that we study is fundamentally different. While the fact to be learned is exogenous in their models, our model features strategic interaction with moral hazard, in which the fact to be learned (i.e., whether the agent is shirking) is endogenous to the choice of monitoring technologies, and in turn, to the principal's design of an incentive scheme by the principal.

2 The Model

2.1 Setup

Time is continuous and infinite. There is a principal ("he") and an agent ("she"). Both are risk neutral. The principal has a discount rate $r > 0$ and unlimited access to capital. The agent has a discount rate $\rho > r$ and is protected by limited liability. The principal owns a project that requires operation by the agent, which involves an action $a_t \in [0, 1]$ taken by the agent. The action can be understood as the level of shirking. If action a_t is taken at instant t, in the period $[t, t + dt]$, the agent enjoys a private benefit of $\lambda \cdot a_t dt$, while the benefit to the principal is $z \cdot (1 - a_t)\, dt > 0$. The principal can terminate the project at any time, and the project then generates a payoff of zero for both players.

Here, we interpret z as the latent progress of a project or the reputation of an entity that is lost without the agent's due diligence and is not discernible immediately. Therefore, contracts cannot be made contingent on whether z is accrued. We interpret z this way for two reasons. First, it captures the reality, mentioned in the Introduction, that the agent's hidden actions are frequently not reflected in existing indicators, such as current output, sales or stock prices. This is because, the outcome of such actions (e.g., the accumulation of intangible assets or reputation) may be realized only in the long run. Second, it separates the role of output as a component of physical payoff from that as a given performance indicator; the latter being well studied in the literature. This allows us to focus on the principal's active acquisition of information regarding the agent's action. For ease of presentation, we hereafter refer to z as the "synergy" (between the principal and the agent).

To model the principal's capacity-allocation decision, we assume that at each instant the principal can choose how to allocate his μ units of monitoring capacity between "carrot-based search" ("C-search" hereafter) and "stick-based search" ("S-search" hereafter); i.e., to seek one of two types of evidence as the basis for reward and penalty. The receipt of C-evidence confirms the effort exerted by the agent, while the receipt of S-evidence contradicts it.

Specifically, if the principal allocates a fraction $\alpha_t \in [0, \bar{\alpha}]$ of his μ units of monitoring capacity to seeking S-evidence and the remaining $1 - \alpha_t$ fraction of his monitoring capacity for the C-evidence, he receives S-evidence at the arrival rate $\mu \cdot \alpha_t \cdot a_t$, and C-evidence at the arrival rate $\mu \cdot (1 - \alpha_t) \cdot (1 - a_t)$. Hence, the agent's chance of being caught shirking is proportional to a_t, the level of shirking, and $\mu \cdot \alpha_t$, the capacity allocated to monitoring shirking. Intuitively, if the agent does not shirk, no evidence of shirking exists, and the principal cannot find S-evidence no matter how much capacity is allocated to seeking such evidence; if the principal allocates no capacity to monitor shirking, he receives no S-evidence regardless of the agent's level of shirking. The arrival rate of C-evidence can be interpreted similarly. More specifically, the cumulative number of arrivals of S-evidence, Y_1, and that of C-evidence, Y_0, satisfy

$$dY_{1,t} = \begin{cases} 1, & \text{with probability } \mu \alpha_t a_t dt \\ 0, & \text{otherwise} \end{cases},$$

and

$$dY_{0,t} = \begin{cases} 1, & \text{with probability } \mu (1 - \alpha_t)(1 - a_t)\, dt \\ 0, & \text{otherwise} \end{cases},$$

respectively. To save the notation, we write $Y = (Y_0, Y_1)$.

It is worth noting that upper bound $\bar{\alpha}$ measures the flexibility of the principal in allocating his capacity across C-search and S-search. To highlight the role of this flexibility, we assume $\bar{\alpha}$ to be close to 1. However, if $\bar{\alpha} = 1$, we show in the Appendix that the optimal reward to the agent upon the arrival of C-evidence would be infinity with positive probability, and thus we exclude this case in the text.[3] Formally, we assume that

Assumption 1 $\bar{\alpha} \in [1 - \frac{\rho}{\mu}, 1)$.

We also assume that the principal is more patient than the agent, and that the principal has enough capacity to monitor the agent in this contractual relationship.

Assumption 2 $r < \rho < \mu$.

[3] See the discussion following Equation (14) for details.

Moreover, we assume that z is large enough so that shirking (action 1) is inefficient even taking into account the agent's private benefit. Then it is optimal for the principal to always implement $a_t = 0$ and thus it is without loss of generality to focus on such contracts.[4]

Assumption 3 $z > \lambda > 0$.

A contract X specifies the recommended action a taken by the agent, the monitoring scheme α,[5] the cumulative payment I to the agent and the time of termination τ as functions of the history of past evidence. As mentioned before, we focus on contracts that implement $a_t = 0$ for all t, so that we suppress a and write $X = (\alpha, I, \tau)$.

Given the contract X and an action process a, the expected discounted utility of the agent is

$$\mathbb{E}^a \left[\int_0^\tau e^{-\rho t} \left(dI_t + \lambda a_t dt \right) \right],$$

and that of the principal is

$$\mathbb{E}^a \left[\int_0^\tau e^{-rt} \left(z \left(1 - a_t\right) dt - dI_t \right) \right]. \tag{1}$$

For notational convenience, we hereafter suppress all time subscripts when no confusion can be caused.

[4] This is formally established in Section 7.4 in the Appendix.

[5] The capacity μ in our model should be understood generically as resources available to the principal for monitoring the agent. In reality, this corresponds to the total budget for hiring quality control team, installing call recorders or surveillance cameras, etc. By including the monitoring scheme α (i.e., the allocation of capacity) in the contract, we are studying the benchmark in which the focus of the evaluation of the agent's performance at different stages of the contractual relationship is explicitly specified at the beginning, and is implemented throughout the relationship. This benchmark is realistic under many circumstances, especially for firms, organizations or bureaucratic systems that specify the details of their routine monitoring practice for employees at different positions with different seniority in contracts, charters or code of conducts. Situations where the principal cannot commit to a monitoring scheme is also realistic and worth studying, but beyond the scope of this paper.

While contracts involving randomization are of theoretical interest, they are typically not practical in reality. Therefore, we postpone the discussion of public randomization to Section 5, and consider only deterministic contracts for the rest of this paper unless otherwise mentioned.

2.2 Incentive Compatibility and Limited Liability

To characterize the incentive compatibility condition, we rely on martingale techniques similar to those introduced by (Sannikov 2008). When choosing her action at time t, the agent considers how it will affect her continuation value, defined as

$$w_t(X,a) = \mathbb{E}^a\left[\int_t^\tau e^{-\rho u}\left(dI_u + \lambda a_u du\right)\Big|\,\mathcal{F}_t\right]1_{\{t<\tau\}},$$

where $\{\mathcal{F}_t\}$ is the filtration generated by Y. Martingale representation theorem yields the following lemma.

Lemma 2.1 *For any contract X that implements $a_t = 0$ for all $t \le \tau$, there exist predictable processes (β_0, β_1) such that w_t evolves before termination $(t \le \tau)$ as*

$$dw_t = \rho w_t dt - dI_t + \beta_{0,t}\left[dY_{0,t} - \mu\left(1-\alpha_t\right)dt\right] - \beta_{1,t}dY_{1,t}\ . \tag{2}$$

The contract is incentive compatible if and only if

$$\mu\alpha_t\beta_{1,t} + \mu(1-\alpha_t)\beta_{0,t} \ge \lambda\ . \tag{IC}$$

And the contract satisfies the limited liability constraint of the agent if and only if

$$\beta_{1,t} \le w_t \tag{3}$$

and

$$\beta_{0,t} + w_t \ge 0\ . \tag{4}$$

The proofs of this lemma and of all the other lemmas and propositions are relegated to the Appendix unless otherwise specified. Intuitively, β_0 refers to the reward to the agent upon the receipt of C-evidence, and β_1 refers to the punishment to her upon the receipt of S-evidence. Hereafter, we refer to β_0 as "carrots", and β_1 as "sticks". Inequality (IC) highlights the key feature of our model. Its left-hand side consists of the two instruments, C-search and S-search, that the principal uses to incentivize the agent, which must sum to at least λ, the agent's private benefit from shirking. The principal can choose not only the allocation of his monitoring capacity α, but also β_0 and β_1, the carrots and sticks associated.

Two limited liability constraints in Lemma 2.1 restrict the magnitudes of reward and punishment. Inequality (3) requires that sticks should be no more than the whole stake promised to the agent. The other constraint (4) says that carrots plus the stake already promised to the agent has to be non-negative, which will be shown slack.

3 Basic Properties of the Optimal Contract

This section provides a heuristic derivation of some basic properties of the optimal contract. Theorem 3.1 at the end of this section verifies that this contract is indeed optimal.

Let $B(w)$ denote the principal's value function. We have the Hamilton–Jacobi–Bellman (HJB) equation in the continuation region ($t < \tau$)

$$rB(w) = \max_{\alpha,\beta_0,\beta_1} z+(1-\alpha)\mu\left[B(w+\beta_0)-B(w)\right]+\left[\rho w-\beta_0\mu(1-\alpha)\right]B'(w) \ , \tag{5}$$

subject to

$$\mu\alpha\beta_1+\mu(1-\alpha)\beta_0 \geq \lambda \ ; \tag{IC}$$

$$\beta_1 \leq w \ ; \tag{6}$$

$$\beta_0 + w \geq 0 \ ; \tag{7}$$

and

$$\alpha \in [0, \bar{\alpha}] \ . \tag{8}$$

The left-hand side of equation (5) is the principal's expected flow of value. The first term on the right-hand side, z, is the flow of synergy. The second term is due to the carrots β_0 he gives to the agent if C-evidence is obtained, which happens with probability $(1-\alpha)\mu dt$ conditional on $a = 0$ being implemented from t to $t + dt$. The third term arises from the drift of w, where ρw is the rate at which interest accrues, and $-\beta_0\mu(1-\alpha)$ is the flip side of carrots due to promise keeping: if there is no C-evidence, the principal reduces the agent's continuation value at this rate to balance against carrots, so that the continuation value w_t net of a drift $\rho w_t dt$ is a martingale, and thus the contract does deliver w_t in expectation to the agent.

Note that there is no term in equation (5) that corresponds to sticks (i.e., no term containing β_1), because S-evidence is never obtained if the agent follows the contract and takes $a = 0$ at each instant. In this sense, sticks serve only as an off-equilibrium threat. Therefore, the limited liability constraint (6) must be binding: If S-evidence were obtained, the principal would maximize the penalty by terminating the project and confiscating the whole stake w promised to the agent.

Notationally, superscript * hereafter denotes items in the optimal contract.

Property 1 $\beta_1^*(w) = w$.

Instead of $B(w)$, it is equivalent but more convenient to continue our analysis based on $V(w) = B(w) + w$, the sum of the principal's value function and the agent's continuation value, or their joint surplus. Equation (5) then becomes

$$rV(w) = \max_{\alpha,\beta_0} z + [\rho w - \beta_0\mu(1-\alpha)]V'(w) + (1-\alpha)\mu[V(w+\beta_0) - V(w)] - (\rho - r)w \ . \tag{9}$$

Next, since $r < \rho$, we guess and later verify that there is a payout boundary $\bar{w}$ as standard in existing dynamic contracting models, e.g., (DeMarzo and Sannikov 2006) and (Biais, Mariotti, Rochet, and Villeneuve 2010). If $w >$

$\bar{w}$, the principal will simply pay $dI = w - \bar{w}$ immediately and reduce the continuation value to $\bar{w}$. Otherwise, the principal will use backloading; i.e., wait for the agent's continuation value w to increase instead of paying her immediately (i.e., $dI = 0$). By construction, $V(\bar{w} + \beta_0) = V(\bar{w})$, so that when $w = \bar{w}$, the third term on the right-hand side of equation (9) equals zero, and $V'(\bar{w}) = 0$ if it exists. If $V'(\bar{w})$ does not exist; i.e., the left and the right derivatives are not equal, equation (9) is not defined at $w = \bar{w}$, which means that the coefficient in front of $V'(w)$ is zero at $\bar{w}$. Notice that this coefficient is the drift of the continuation value. Hence, when $V'(\bar{w})$ does not exist, $\bar{w}$ is an absorbing payout boundary. As a result, regardless of whether the payout boundary $\bar{w}$ is absorbing or not, the second term in equation (9) must also equal zero when $w = \bar{w}$, so that

$$V(\bar{w}) = \frac{z}{r} - (\rho - r)\frac{\bar{w}}{r} \tag{10}$$

and

$$B(\bar{w}) = \frac{z}{r} - \frac{\rho}{r}\bar{w} \ . \tag{11}$$

Moreover, we must have $\bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}}$. If not, then once the continuation value reaches $\bar{w} > \frac{\lambda}{\rho+\mu\bar{\alpha}}$, the principal could always incentivize the agent with the following contract: paying out $\bar{w} - \frac{\lambda}{\rho+\mu\bar{\alpha}}$ immediately to reduce the agent's continuation value to $\frac{\lambda}{\rho+\mu\bar{\alpha}}$; setting $\alpha = \bar{\alpha}$, $\beta_1 = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ and $\beta_0 = \frac{\rho\lambda}{\mu(1-\bar{\alpha})(\rho+\mu\bar{\alpha})}$, so that (IC) is binding, and that $\beta_0\mu(1-\bar{\alpha}) = \rho\frac{\lambda}{\rho+\mu\bar{\alpha}}$; i.e., the drift of the agent's continuation value is zero, and thus $w = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is an absorbing state.[6] Then the principal's payoff becomes

$$\frac{z}{r} - \frac{\rho}{r} \cdot \frac{\lambda}{\rho + \mu\bar{\alpha}} - (\bar{w} - \frac{\lambda}{\rho + \mu\bar{\alpha}}) > \frac{z}{r} - \frac{\rho}{r}\bar{w} = B(\bar{w}) \ ,$$

where the inequality follows $\bar{w} > \frac{\lambda}{\rho+\mu\bar{\alpha}}$, contradicting the optimality of $B(\bar{w})$. As a standard result in this literature, the optimality of B implies $B'(w) > -1$

[6] More precisely, under this contract, once $w = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, the continuation value never drifts away and the agent receives discrete payments of $\beta_0 = \frac{\rho\lambda}{\mu(1-\bar{\alpha})(\rho+\mu\bar{\alpha})}$ at the arrival rate $\mu(1-\bar{\alpha})$ forever.

for $w < \bar{w}$ and $B'(w) = -1$ for $w \geq \bar{w}$. Then by definition, $V'(w) > 0$ for $w < \bar{w}$ and $V'(w) = 0$ for $w \geq \bar{w}$. We summarize these results in the following property.

Property 2 *There exists a* $\bar{w} \in (0, \frac{\lambda}{\rho+\mu\bar{\alpha}}]$ *such that i)* $dI^* = (w - \bar{w})^+$ *; ii)* V *is increasing in* $[0, \bar{w}]$*; iii) if* $w \geq \bar{w}$*,*

$$V(w) = z/r - (\rho - r)\bar{w}/r \ ; \tag{12}$$

and iv) either $V'(\bar{w}) = 0$*, or* $\rho\bar{w} - \mu(1 - \alpha^*(\bar{w}))\beta_0^*(\bar{w})$*, the drift at* $w = \bar{w}$*, is* 0*.*

Together with Assumption 1 and Property 1, we have $\beta_1^*(w) = w < \bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}} < \lambda/\mu$ for $w < \bar{w}$; i.e., by the (IC) constraint, sticks alone are not sufficient to incentivize the agent to work. Moreover, (IC) and Property 1 imply that $w\alpha^* + \beta_0^*(1 - \alpha^*) \geq \lambda/\mu$, thus $\beta_0^*(w) \geq \lambda/\mu \geq \frac{\lambda}{\rho+\mu\bar{\alpha}} \geq \bar{w}$ for $w < \bar{w}$. This, together with Property 2, implies

Property 3 $w + \beta_0^* \geq \bar{w}$ *for all* $w < \bar{w}$*.*

That is, a single piece of C-evidence suffices to make the continuation value w jump to the payout region $[\bar{w}, +\infty)$, so that $V(w + \beta_0^*) = V(\bar{w})$; i.e., β_0^*, the carrots, raises their joint surplus only from $V(w)$ to $V(\bar{w})$, and the remaining reward, $\beta_0^* - (\bar{w} - w)$, is an immediate transfer from the principal to the agent and has no impact on their joint surplus. Also, the limited liability constraint (7) slacks as conjectured.

Property 3 plays a crucial role in the derivation of the optimal contract, given that the value function V may not always be concave.[7] To see this, note that according to Property 3, equation (9) becomes

$$rV(w) = \max_{\beta_0,\alpha} z + [\rho w - \beta_0\mu(1 - \alpha)]\, V'(w) + (1 - \alpha)\,\mu[V(\bar{w}) - V(w)] - (\rho - r)w \ , \tag{13}$$

[7] This possibility is discussed in Subsection 4.2. We also concavify the value function via public randomization in Section 5.

whose right-hand side is always decreasing in β_0. This has two important implications. First, it indicates the advantage of using S-search rather than C-search, regardless of the concavity of V. In equilibrium, S-evidence is never obtained, and thus S-search incentivizes the agent without causing variation in her continuation value w. But if C-search is used (i.e., $\alpha < 1$), C-evidence is obtained in equilibrium and generates variation in w. Property 3 implies that effectively, the upward jump in w upon the receipt of C-evidence is always $\bar{w} - w$ (after the bonus payment), which is independent of α and β_0. But the magnitude of the downward drift of w in the absence of C-evidence, $\beta_0\mu(1-\alpha)$, is increasing in both the capacity allocated to C-search, $\mu(1 - \alpha)$, and the associated carrots, β_0. Therefore, the more the principal resorts to C-search, the more adverse variation in w is generated, making it detrimental relative to sticks.

Second, the fact that the right-hand side of equation (13) is decreasing in β_0 implies a binding incentive compatibility constraint (IC) in the no-payment region $[0, \bar{w}]$, i.e.;

Property 4 $\mu\left[\alpha^* w + (1 - \alpha^*)\beta_0^*\right] = \lambda$.

The incentive compatibility constraint (IC) plays a central role in this model. Property 4 establishes that the combination of C-search and S-search should be just enough to overcome the agent's private benefit from shirking.

Note that the principal still has two degrees of freedom to adjust the sensitivities of the agent's continuation value to news reflecting her actions. As mentioned in the literature review, this contrasts with the counterpart in models without choice among multiple performance indicators; e.g., in (Sannikov 2008) and (Biais, Mariotti, Rochet, and Villeneuve 2010), where there is no such degree of freedom.

Now we are ready to derive the central piece of the model — the optimal allocation of monitoring capacity, α, and the optimal carrots, β_0, in the no-payment region $[0, \bar{w}]$. Given Properties 1 and 4,

$$\beta_0^* = \frac{\lambda - \mu\alpha^* w}{\mu\,(1 - \alpha^*)}. \tag{14}$$

Equation (14) highlights the substitution between the capacity allocated to C-search, $1-\alpha$, and the associated carrots, β_0, which is peculiar to our setup with flexibility in monitoring practice. The more capacity is allocated to C-search, the higher is the probability of obtaining C-evidence that confirms the agent's effort, and thus less reward is needed to incentivize the agent. Conversely, higher carrots provide a stronger incentive for the agent, and thus reduce the reliance of the principal on obtaining C-evidence, enabling him to utilize S-search. Note that $\beta_0^* \to \infty$ as $\alpha^* \to 1$, and thus we assume $\bar{\alpha} < 1$ in Assumption 1 to preclude this situation.

In the no-payment region, we have $dI = 0$ by definition and $V(w + \beta_0) = V(\bar{w})$ from Property 2. As a result of equation (14), the HJB equation (9) becomes

$$rV(w) = \max_{\alpha \in [0,\bar{\alpha}]} z - (\rho - r)w + (1-\alpha)\mu[V(\bar{w}) - V(w)] + (\rho w - \lambda + \mu\alpha w)V'(w). \tag{15}$$

Notice that α affects the right-hand side of equation (15) through the last two terms. As explained before, the third term reflects its impact through carrots; i.e., raising α reduces the arrival rate of C-evidence and that of the contingent increment $V(\bar{w}) - V(w)$ in their joint surplus. This in turn reduces the expected instantaneous joint surplus $(1-\alpha)\mu[V(\bar{w}) - V(w)]$. The impact is linear in α, and the marginal impact is $-\mu[V(\bar{w}) - V(w)]$, whose absolute value decreases monotonically with w.

The last term on the right-hand side of equation (15) reflects the impact of α through the flip side of carrots; i.e., a lower arrival rate of C-evidence also reduces the downward drift of the agent's continuation value w due to promise keeping.[8] This increases the expected instantaneous joint surplus $(\rho w - \lambda + \mu\alpha w)V'(w)$. This effect is also linear in α, with a marginal impact $\mu w V'(w)$, which could be non-monotonic in w. Since the total impact of α is

[8] Note that $\rho w - \lambda + \mu\alpha w \leq 0$ since $\bar{w} \leq \frac{\lambda}{\rho + \mu\bar{\alpha}}$. Raising α thus reduces the magnitude of the downward drift.

linear, with marginal impact

$$\mu\left[wV^{'}(w)+V(w)-V(\bar{w})\right],\tag{16}$$

we have the following corner solution.

Property 5 *If* $wV^{'}(w)+V(w)<V(\bar{w})$*, then* $\alpha^{*}=0$ *and* $\beta_{0}^{*}=\lambda/\mu$*;*
If $wV^{'}(w)+V(w)=V(\bar{w})$*, then* $\alpha^{*}\in[0,\bar{\alpha}]$ *and* $\beta_{0}^{*}=\frac{\lambda-\mu\alpha^{*}w}{\mu(1-\alpha^{*})}$*;*
If $wV^{'}(w)+V(w)>V(\bar{w})$*, then* $\alpha^{*}=\bar{\alpha}$ *and* $\beta_{0}^{*}=\frac{\lambda-\mu\bar{\alpha}w}{\mu(1-\bar{\alpha})}$.

The following theorem verifies that the contract that we derive is indeed optimal.

Theorem 3.1 *Under Assumptions 1, 2 and 3, the solution* V *to HJB equation* (9) *is the principal and the agent's joint surplus under the optimal contract. Moreover, the optimal contract is characterized by Property 5.*

4 The Role of Flexible Monitoring

This section highlights the critical role of flexible monitoring, which is central to this paper. Section 4.1 shows that such flexibility is indeed utilized by and thus valuable to the principal. Section 4.2 further articulates that such flexibility allows a long-term contractual relationship that perpetuates the agent's effort with positive probability when the synergy, z, is sufficiently large, and that the value function is convex in the vicinity of the payout boundary $\bar{w}$ if and only if such perpetuation is optimal. Section 4.3 summarizes these results with a graphic illustration using the narrative of career path and provides a few empirically plausible predictions.

4.1 Flexibility in Monitoring is Utilized

Property 5 establishes that other than in knife-edge cases, the optimal monitoring capacity allocated to S-search, α^{*}, is either 0 or $\bar{\alpha}$.[9] This subsection

[9] Lemma 7.2 in the Appendix shows that there does not exist an interval of continuation values such that the principal is indifferent between 0 and $\bar{\alpha}$. Thus the knife-edge cases are

further establishes that an optimal contract necessarily involves both possibilities. Specifically, Proposition 4.1 establishes that $\alpha^*(w) = 0$ when the agent's continuation value w is close to 0, and $\alpha^*(w) = \bar{\alpha}$ when w is close to the payout boundary $\bar{w}$. This indicates that flexibility in allocating monitoring capacity between C-search and S-search allows the principal to incentivize the agent differently at different stages of her career, and is thus valuable to the principal.

Proposition 4.1 *There exists a $\hat{w}_0 \in (0, \bar{w})$ and a $\hat{w}_{\bar{\alpha}} \in [\hat{w}_0, \bar{w})$, such that $\alpha^*(w) = 0$ and $\beta_0^*(w) = \lambda/\mu$ if $w \in (0, \hat{w}_0)$, and that $\alpha^*(w) = \bar{\alpha}$ and $\beta_0^*(w) = \frac{\lambda - \mu\bar{\alpha}w}{\mu(1-\bar{\alpha})}$ if $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$.*

From Property 5, the optimal contract only involves $\alpha = 0$ and $\alpha = \bar{\alpha}$ except for the knife-edge case featuring indifference. From equation (15) we know that for each $w \in (0, \bar{w})$, either $\alpha = 0$ and

$$rV(w) = z + [\rho w - \lambda]V'(w) + \mu[V(\bar{w}) - V(w)] - (\rho - r)w \ , \tag{17}$$

or $\alpha = \bar{\alpha}$ and

$$rV(w) = z + (1-\bar{\alpha})\mu[V(\bar{w}) - V(w)] + [\rho w - \lambda + \mu\bar{\alpha}w]V'(w) - (\rho - r)w \ . \tag{18}$$

Both equations can be solved in closed form, and interested readers are referred to the Appendix. It can be verified that $V'(0)$ is finite. This implies $0 \cdot V'(0) + V(0) = 0 < V(\bar{w})$, and by continuity, there is a neighborhood of $w = 0$ such that $wV'(w) + V(w) < V(\bar{w})$. Thus, by Property 5, the principal relies completely on C-search when the agent's continuation value w is low. The statement for $(\hat{w}_{\bar{\alpha}}, \bar{w})$ can also be proved with the closed-form solutions similarly.

Intuitively, when the agent's continuation value w is low, the principal should not rely on S-search at all, because the agent has little to lose even if she is known to have shirked. Relying on C-search, on the other hand, also

non-generic.

maximizes the chance of obtaining C-evidence (i.e., evidence confirming the agent's effort). This helps the principal quickly raise the agent's "skin in the game", which makes S-search (which is costless to the principal) more effective in the future, and pushes the project away from termination (which is socially inefficient). When the agent's continuation value w is higher, the principal can impose a large penalty if he obtains S-evidence. Since such a penalty is just an off-equilibrium threat, making S-search less costly than C-search, the principal should rely on S-search as much as possible.

The flexibility of combining C-search and S-search allows the principal to exploit their respective advantages. On one hand, C-search generates greater variation than S-search in the agent's continuation value, and is thus less advantageous to the principal. This is because, given that the agent does work, no S-evidence would be obtained, and thus, no adjustment to the agent's continuation value would be required. However, in equilibrium, C-evidence would be obtained, which would necessarily involve a reward and the downward adjustment to the agent's continuation value in the absence of C-evidence. On the other hand, a sufficiently high continuation value is required as the agent's skin in the game for sticks alone to be an effective incentive device, whereas the effectiveness of C-search does not depend on the agent's continuation value. Moreover, even if S-search could work alone, a high continuation value for the agent has to be maintained, which involves interest expenditure for the principal, making S-search less advantageous than C-search. This tradeoff between C-search and S-search induces the principal to rely only on C-search when w is low, and on S-search, as much as possible, when w is high.

Concerning the carrots, β_0, recall that the right-hand side of equation (13) is decreasing in β_0, since an increase in β_0 makes the drift of the agent's continuation value, $\rho w - \beta_0 \mu (1-\alpha)$, more negative due to promise keeping, and thus makes the project more prone to termination. Hence, given the optimal capacity allocation α^*, β_0^* should be set as low as possible — such that the incentive compatibility constraint (IC) is binding. Thus, for agents facing $\alpha^* = 0$, including those with $w \in (0, \hat{w}_0)$, we have $\beta_0^*(w) = \lambda/\mu$, and the resulting drift of w is $\rho w - \lambda < 0$. For agents facing $\alpha^* = \bar{\alpha}$, including those with $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$, we have $\beta_0^*(w) = \frac{\lambda - \mu\bar{\alpha}w}{\mu(1-\bar{\alpha})}$, and the resulting drift of w is $\rho w - \lambda + \mu\bar{\alpha}w \leq 0$.[10]

Note first that $\beta_0^*(w)$ is constant in the region of $\alpha^*(w) = 0$, but is decreasing in the region of $\alpha^* = \bar{\alpha}$. This is because in the latter case, sticks increase with w, partially substituting carrots that are required by the incentive compatibility constraint (IC). Second, $\beta_0^*(w)$ features an upward jump when α^* switches from 0 to $\bar{\alpha}$. To see this, notice the fact that any switching point $w < \frac{\lambda}{\rho+\mu\bar{\alpha}} \leq \frac{\lambda}{\mu}$ implies that the size of the upward jump is $\frac{\lambda-\mu\bar{\alpha}w}{\mu(1-\bar{\alpha})} - \frac{\lambda}{\mu} > \frac{\lambda-\mu\bar{\alpha}\cdot\frac{\lambda}{\mu}}{\mu(1-\bar{\alpha})} - \frac{\lambda}{\mu} = 0$. Third, the drift of w increases (i.e., becomes less negative) with w, due to the interest accrued (i.e., due to the term ρw) and the increasing reliance on S-search in lieu of C-search (i.e., due to the term $\mu\bar{\alpha}w$). Lastly, the drift of w is negative, which moves w towards 0, the termination boundary, unless w reaches the payout boundary $\bar{w}$ and $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, where the drift is zero; i.e., the project and the agent's effort are perpetuated. Section 4.2 characterizes when such perpetuation is optimal.

4.2 Possibility of Perpetuating the Agent's Effort

This subsection discusses whether the optimal contract involves the perpetuation of the agent's effort with positive probability. Mathematically, this refers to whether the payout boundary $\bar{w}$ is an absorbing state. We show that this is related to the (local) convexity of the value function, which is in turn determined by the flexibility of the principal's capacity allocation as captured by $\bar{\alpha}$, and by the magnitude of the synergy z to that of the agent's private benefit from shirking, λ. Specifically, 1) $\bar{w}$ is absorbing if and only if $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$; 2) $\bar{w}$ is absorbing if and only if the value function V is not universally concave[11]. More precisely, $\bar{w}$ is absorbing if and only if V is convex in $(\hat{w}_{\bar{\alpha}}, \bar{w})$ given by Proposition 4.1; and 3) $\bar{w}$ is absorbing if and only if $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$ and z is sufficiently large.

Again, the role of flexibility in capacity allocation is worth highlighting.

[10] This is because $w \leq \bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}}$.

[11] Recall from Section 2 that we will discuss public randomization in Section 5 and preclude it in the rest of the paper unless otherwise mentioned.

We show that without such flexibility, perpetuation of the agent's effort is not optimal.

Recall from Property 2 that $\bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}}$. We have in addition

Lemma 4.1 *$\bar{w}$ is absorbing if and only if $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$.*

Proof. First consider the "if" statement. If $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, we show that the following strategy is feasible and optimal, and makes $\bar{w}$ absorbing: $\alpha = \bar{\alpha}$, $\beta_0 = \frac{\rho\lambda}{\mu(1-\bar{\alpha})(\rho+\mu\bar{\alpha})}$ and $\beta_1 = \bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$. Feasibility results from the binding (IC) constraint. To see why $\bar{w}$ is absorbing, when $w = \bar{w}$, the positive component of the drift of the agent's continuation value due to accrued interest is $\rho\bar{w}dt = \frac{\rho\lambda}{\rho+\mu\bar{\alpha}}dt$, and the negative component as the flip side of carrots is $\mu(1-\bar{\alpha})\beta_0 dt$, which also equals $\frac{\rho\lambda}{\rho+\mu\bar{\alpha}}dt$, so that w remains constant when there is no C-evidence, and when it is obtained, the whole reward β_0^* is paid out immediately so that w remains at $\frac{\lambda}{\rho+\mu\bar{\alpha}}$.

To see the optimality of this strategy, observe that the principal's expected payoff at $w = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is $\mathbb{E}(\int_0^{+\infty} ze^{-rt}dt - \beta_0 \int_0^{+\infty} e^{-rt}dY_{0,t})$. Since $Y_{0,t} - \mu(1-\bar{\alpha})t$ is a martingale,

$$\mathbb{E}(\int_0^{+\infty} ze^{-rt}dt - \beta_0 \int_0^{+\infty} e^{-rt}dY_{0,t}) = \frac{z}{r} - \frac{\beta_0\mu(1-\bar{\alpha})}{r} = \frac{z}{r} - \frac{\rho}{r} \cdot \frac{\lambda}{\rho+\mu\bar{\alpha}} \ .$$

Thus, the expected joint surplus is

$$\frac{z}{r} - \frac{\rho}{r} \cdot \frac{\lambda}{\rho+\mu\bar{\alpha}} + \bar{w} = \frac{z}{r} - \frac{\rho - r}{r} \cdot \bar{w} \ .$$

From equation (10), this strategy achieves the optimal joint surplus at the payout boundary.

Now consider the "only if" statement. From Property 2, it suffices to show that any $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$ cannot be absorbing. Any contract respecting the (IC) constraint satisfies

$$\beta_0\mu(1-\bar{\alpha}) \geq \lambda - \bar{w}\mu\bar{\alpha} > \rho \cdot \frac{\lambda}{\rho+\mu\bar{\alpha}} > \rho\bar{w} \ .$$

Thus, when there is no C-evidence, the agent's continuation value always has a downward drift term $\rho\bar{w} - \beta_0\mu(1 - \bar{\alpha}) < 0$. This implies that the payout boundary $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is reflective. ■

Notice the role of flexibility in capacity allocation here. If $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, the way for the principal to perpetuate the agent's effort is to set $\alpha = \bar{\alpha}$; i.e., to rely on S-search as much as possible. But if $w_0 < \hat{w}_0$, such a monitoring scheme is not viable since the agent has too little to lose if caught shirking. To avoid inefficient termination, the principal must first rely on C-search to build up the agent's skin in the game while keeping her working, and then switch to stick-dominant mode when the continuation value is high enough. This approach is impossible without flexibility in capacity allocation.

As the main proposition of this subsection, Proposition 4.2 further establishes the connection between the possibility of perpetuating the agent's effort, the (local) convexity of the value function, and the conditions on exogenous parameters.

Proposition 4.2 *Let $\hat{w}_0$ and $\hat{w}_{\bar{\alpha}}$ be given by Proposition 4.1. Then, $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ (i.e., absorbing) if and only if V is convex in $(\hat{w}_{\bar{\alpha}}, \bar{w})$, which holds if and only if $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$ and z is sufficiently large. Moreover, when $\bar{w}$ is absorbing, $\hat{w}_0 = \hat{w}_{\bar{\alpha}}$.*

That is, the agent's effort can be perpetuated if and only if *both* the principal enjoys sufficient flexibility in capacity allocation *and* the synergy of the contractual relationship is large enough. In other words, no matter how large is the synergy, the contractual relationship would terminate in probability one as long as the principal does not have sufficient flexibility of capacity allocation. This again stresses the importance of such flexibility in shaping the optimal contract.

Proposition 4.2 also establishes the equivalence relation between $\bar{w}$ being absorbing and the local convexity of the value function V. Moreover, if the agent's effort could be perpetuated, there is only one switching point $\hat{w}_{\bar{\alpha}} \in (0, \bar{w})$ for capacity allocation; i.e., the optimal $\alpha = 0$ in $(0, \hat{w}_{\bar{\alpha}})$, and $\alpha = \bar{\alpha}$ in $(\hat{w}_{\bar{\alpha}}, \bar{w})$.

We explain the role of flexibility in capacity allocation (i.e., in combining C-search with S-search) in three steps. First, we argue that it is possible for the optimal contract to have an absorbing payout boundary while satisfying (IC) for all w only if the principal is able to combine C-search with S-search (i.e., to have $\alpha \in (0, 1)$). To see this, consider first the situation where only C-search is viable (i.e., α is fixed to 0) as in (Sun and Tian 2017). From Property 4, $\beta_0^*(w) = \lambda/\mu$ for all $w \leq \bar{w}$, and the value function V satisfies equation (17). It is straightforward from (Sun and Tian 2017) that V must be concave, so that $\bar{w}$ is reflective. This reflects the standard incentive versus interest tradeoff in the literature. That is, an increase in w pushes the continuation value away from the termination boundary 0, whose marginal benefit decreases with w, but whose marginal cost, due to an increase in accrued interest, is constant. Such a tradeoff is also featured in (Biais, Mariotti, Rochet, and Villeneuve 2010) and (DeMarzo and Sannikov 2006). In this case, the agent receives a lumpy bonus of $\beta_0 - (\bar{w} - w)$ and a jump of $\bar{w} - w$ in her continuation value upon the receipt of each piece of C-evidence, but her continuation value will still drift downward from $\bar{w}$ until the next piece of C-evidence is obtained, and will eventually reach zero in probability one, resulting in termination of the project. In the opposite situation, where only S-search is viable (i.e., α is fixed to 1), if $w < \lambda/\mu$, it is impossible to satisfy the incentive compatibility constraint (IC), and thus no contract implements the desired action; otherwise, it is optimal to choose an absorbing payout boundary $\bar{w} = \lambda/\mu$ by making a flow payment $\rho\bar{w}$ forever. Therefore, it is the feasibility of combining C-search with S-search that enables the combination of the incentive compatibility of effort for all w from C-search and the absorbing payout boundary from S-search in the optimal contract.

Second, we argue that flexibility in capacity allocation allows the principal to optimally decide whether to make the payout boundary $\bar{w}$ absorbing. On one hand, doing so perpetuates the agent's effort when her continuation value w is high. On the other hand, doing so also leads to the accrual of higher interest from maintaining the continuation value at $\bar{w}$ and a lower arrival rate of C-evidence when w is low. To see this, suppose α is fixed to $\bar{\alpha}$, so that the value function V satisfies equation (18) in $(0, \bar{w}]$. Its closed-form solution implies that V is convex and thus that the payout boundary $\bar{w}$ is absorbing if and only if $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$ and z is sufficiently large. In other words, although the option of an absorbing payout boundary is readily available even if α is fixed to $\bar{\alpha}$, such an option is optimal for the principal if and only if he is able to rely on S-search and the synergy is large enough. This is because, while the absorbing payout boundary makes it possible to permanently avoid inefficient termination, it also requires maintaining a high constant continuation value of $\bar{w}$ as potential sticks, resulting in a high constant flow of interest and a reduction of surplus due to the difference in discount rates between the principal and the agent. Therefore, it is advisable to make the payout boundary absorbing if and only if the continuation value $\bar{w}$ can be sufficiently utilized (i.e., $\bar{\alpha}$ is large) and perpetuating the contractual relation is sufficiently beneficial. Moreover, once the principal has the flexibility in adjusting α, he has another option to avoid termination, which is to set $\alpha = 0$ so as to maximize the arrival rate of C-evidence and upward jumps of w for low values of w (see Proposition 4.1). This makes an absorbing payout boundary even less attractive. Therefore, for values of z not high enough, although it is optimal to make $\bar{w}$ absorbing if α is fixed to $\bar{\alpha}$, it is no longer optimal if α can be flexibly adjusted in $[0, \bar{\alpha}]$.

Lastly, notice the novel feature of our model that the value function V is convex in the vicinity of the payout boundary $\bar{w}$ when $\bar{w}$ is absorbing. While the concavity of V in $(0, \hat{w}_{\bar{\alpha}})$ still reflects the standard incentive versus interest tradeoff, new economic forces come into play in $(\hat{w}_{\bar{\alpha}}, \bar{w})$, where $\alpha^* = \bar{\alpha}$. There, the reliance on S-search reduces the downward drift of the continuation value, $\mu (1 - \alpha) \beta_0$, that balances carrots. This raises the marginal benefit of increasing w without affecting the marginal cost, and thus makes V less concave. Moreover, a fundamental change is brought about when $\bar{w}$ becomes absorbing. In that case, the marginal benefit of increasing w results not only from the fact that w is further away from the inefficient absorbing state 0, but also from the fact that w is closer to the efficient absorbing state $\bar{w}$. The latter fact, together with the constant marginal cost due to accrued interest, makes the marginal benefit increasing instead of decreasing in w and thus the value function V convex in $(\hat{w}_{\bar{\alpha}}, \bar{w})$.

4.3 A Career Path Narrative

Using the narrative of career path, this subsection employs a graphic illustration to summarize the role of flexibility in monitoring practice, which is the core of this paper, and provides a few empirically plausible predictions.

Proposition 4.1 establishes that the optimal monitoring and compensation schemes for junior employees (i.e., agents with continuation value $w \in (0, \hat{w}_0)$) are qualitatively different from those for senior employees (agents with continuation value $w \in (\hat{w}_{\bar{\alpha}}, \bar{w})$). Concerning monitoring schemes, incentives for junior employees are in carrot-only mode (i.e., $\alpha = 0$), since they need to accumulate a cushion against unemployment (i.e., termination) and have little to lose even if caught shirking. Senior employees are instead incentivized in stick-dominant mode (i.e., $\alpha = \bar{\alpha}$), since they have enough skin in the game, and sticks are off-equilibrium penalties, which are less costly than on-equilibrium carrots. Thus, our model predicts that

Prediction 1 *Incentives for junior employees are based mainly on confirmatory evidence of their contribution to their employer, while incentives for senior employees are based mainly on contradictory evidence of their contribution, but are compensated by higher rewards upon the receipt of confirmatory evidence.*

Concerning compensation schemes, as an off-equilibrium threat, sticks are always the whole continuation value w and thus increase with the seniority of employees. For junior employees, carrots are set to the minimum level required to induce effort, and is constantly λ/μ, while carrots for senior employees, $\frac{\lambda-\mu\bar{\alpha}w}{\mu(1-\bar{\alpha})}$, decrease with seniority, since they are replaced by sticks at higher continuation values. For super-senior employees; i.e., agents with continuation value $w > \bar{w}$, their promised stakes are so large that a payment $w-\bar{w}$ to reduce accrued interest is urgent enough to dominate their incentive problems. Hence,

Prediction 2 *The reward for each piece of evidence confirming a contribution to the employer varies little among junior employees, but decreases with* *seniority for senior employees. The penalty for each piece of evidence contradicting a contribution to the employer increases with seniority for both junior and senior employees.*

Concerning the possibility of being fired (i.e., termination), in the absence of C-evidence, the drift of the continuation value of junior employees is $\rho w - \lambda$, and that of senior employees is $\rho w - \lambda + \mu\bar{\alpha}w$. Both are negative unless $w = \bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$; i.e., unless the (senior) employee is permanently hired. They become less negative as the continuation value w increases, for two reasons. First, larger stakes in the game carry a larger interest income. Second, larger stakes also allow for larger sticks and less reliance on C-search, and thus less downward drift in the continuation value to balance in-equilibrium carrots. Therefore,

Prediction 3 *Except for those hired permanently, in the ansence of evidence confirming their contribution, an employee becomes more prone to unemployment, and more so if the employee is less senior.*

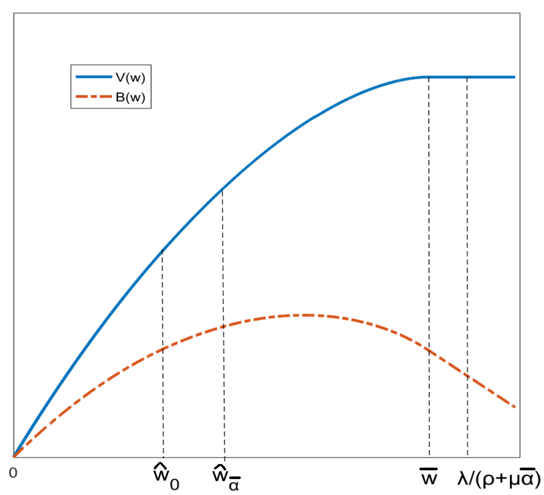

Figure 1: Reflective Payout Boundary $\bar{w}$

When is it possible for employees to be hired permanently? Proposition 4.2 shows that it is the case if and only if both the flexibility in adjusting monitoring schemes and the potential synergy created by employees are sufficiently large.

First, consider the case $\bar{\alpha} \leq \frac{r-\rho+\mu}{2\mu}$ as illustrated in Figure 1; i.e., the flexibility in monitoring is not large enough. The solid blue line corresponds to the value function V (in terms of the joint surplus), and the dash-dotted red line corresponds to the principal's value function $B(w) = V(w) - w$. The value function V is strictly concave in $(0, \hat{w}_0)$, reflecting the standard incentive versus interest tradeoff. V is also strictly concave in $(\hat{w}_{\bar{\alpha}}, \bar{w})$, where the fact that carrots decrease with the agent's stake in the game makes V less concave. However, since $\bar{\alpha}$ is low, the principal does not have enough flexibility to rely on S-search to the extent that he wants, so that $\bar{w}$ is reflective and is thus given by $V'(\bar{w}) = 0$. That is, senior employees who receive carrots still face a downward drift in their promised stakes and thus the risk of being fired. This is the case no matter how large the synergy z is.

Now fix an $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$, so that the principal does have enough flexibility in adjusting the monitoring scheme. If z is small, the synergy is too low to justify perpetuation of the agent's effort, so the optimal contract is qualitatively similar to that of the case $\bar{\alpha} \leq \frac{r-\rho+\mu}{2\mu}$. Once z grows large enough, the optimal contract changes fundamentally, as shown in Figure 2. — Now the principal has both the flexibility and the desire to perpetuate the agent's effort, so now the payout boundary $\bar{w}$ becomes absorbing. That is, the agent is "tenured" once her effort is confirmed by the receipt of C-evidence. In addition, the possibility of completely avoiding termination creates a new marginal benefit of increasing continuation value, and thus makes the value function V convex in $(\hat{w}_{\bar{\alpha}}, \bar{w})$. Moreover, $\hat{w}_0 = \hat{w}_{\bar{\alpha}}$, so that α, the capacity allocated to S-search, only switches once as the agent's continuation value rises from 0 to $\bar{w}$. Thus, we have

Prediction 4 *Employers offer permanent positions if and only if both their flexibility in adjusting monitoring schemes and the potential synergy created by employees are sufficiently large.*

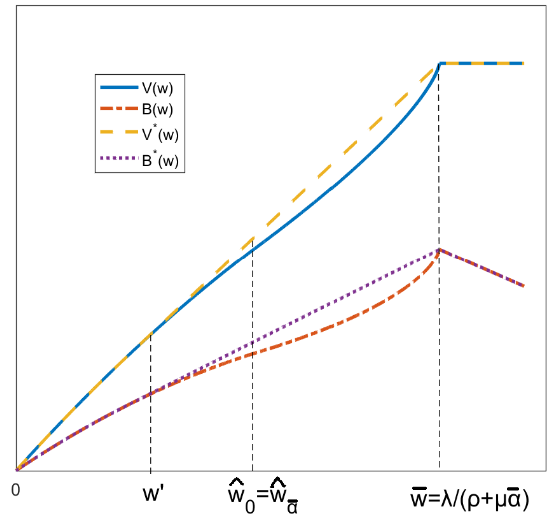

Figure 2: Absorbing Payout Boundary $\bar{w}$

5 Public Randomization

So far, we have been focusing on deterministic contracts, on the basis that random contracts are of little practical relevance in reality. This is also theoretically without loss of generality if the resulting value function is globally concave as in the case illustrated in Figure 1 and as in most models in the literature. However, as established in Proposition 4.2, our value function is convex in the vicinity of the payout boundary $\bar{w}$ if it is absorbing (Figure 2). For this situation, this section discusses the extension in which public randomization of the following form is allowed. At time 0, in addition to starting the contractual relationship with a deterministic continuation value w_0, the principal can choose a mean-preserving spread of w_0 as the basis for random contracts, but no further randomization is allowed for $t > 0$. Since $B = V - w$, and the linear term has no effect on the concavification operation, we can work with the joint surplus function V without loss of generality.

Proposition 5.1 *With public randomization, the principal's value function is $B^* = V^* - w$, where V^* is the concavification of V.*

Proof. Proposition 4.2 establishes that when V is not globally concave, we must have $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, and $V(\bar{w})$ is uniquely determined by Property 2. In addition, V is concave in $(0, \hat{w})$ and convex in $(\hat{w}, \bar{w})$, where $\hat{w} \equiv \hat{w}_0 = \hat{w}_{\bar{\alpha}}$. Therefore, the concavification of V must be over $\bar{w}$ and some $w' \in (0, \hat{w})$ as shown with the yellow broken line in Figure 2.[12]

We check that the values of non-randomized states are not changed. First, $V(\bar{w})$ does not change because $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is absorbing and its value does not depend on the values of other states. For $w \in (0, w')$, notice that the continuation value may only drift downward or jump upward over $\bar{w}$. Since $V(\bar{w})$ remains the same and $V^* = V$ for $w \in (0, w')$, the values of these states satisfy the same HJB equation and thus remain the same. ■

6 Conclusion

This paper studies a continuous-time moral hazard problem in which the principal can flexibly combine C-search with S-search to incentivize the agent. That is, he can flexibly allocate his limited monitoring capacity between confirmatory and contradictory evidence concerning the agent's effort as the basis for reward and punishment. We find that such flexibility generates rich dynamics, which differ qualitatively from the situation where only one of the two methods is feasible. When the agent has little skin in the game, the principal resorts only to C-search; when the agent has a large skin in the game, the principal instead assigns the highest possible weight to S-search. Moreover, only with such flexibility can the agent's effort be perpetuated with positive probability when the agent is less patient than the principal.

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[12]The purple dotted line in Figure 2 illustrates the corresponding concavification B^* of the principal's value function B.

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7 Appendix

7.1 Proofs in Section 2

7.1.1 Proof of Lemma 2.1

Proof. The proof is a standard application of the martingale representation theorem. For any given contract $X = (\alpha, I, \tau)$ and effort process a, define

$$M_t^{1,a} = Y_t^1 - \mu \int_0^t \alpha_s a_s ds$$

and

$$M_t^{0,a} = Y_t^0 - \mu \int_0^t (1-\alpha_s)(1-a_s)ds \ .$$

If the agent follows the effort process a, her lifetime expected payoff, conditional on information at time t, is

$$U_t = \int_0^{t\wedge\tau} e^{-\rho s}(dI_s + \lambda a_s ds) + e^{-\rho t} W_t \ .$$

Let $\tilde{a}$ be an arbitrary effort process. Let $\tilde{U}_t$ denote the agent's lifetime expected payoff conditional on information at time t if she follows $\tilde{a}$ until time t and then reverts to a. Then by the martingale representation theorem, U_t can be written as

$$U_t = U_0 - \int_0^{t\wedge\tau} e^{-\rho s}\beta_{1,s} dM_s^{1,a} + \int_0^{t\wedge\tau} e^{-\rho s}\beta_{0,s} dM_s^{0,a}$$

For each $t \geq 0$,

$$\begin{aligned}
\tilde{U}_t =& U_t + \int_0^{t\wedge\tau} e^{-\rho s}\lambda(\tilde{a}_s - a_s)ds \\
=& U_0 - \int_0^{t\wedge\tau} e^{-\rho s}\beta_{1,s} dM_s^{1,a} + \int_0^{t\wedge\tau} e^{-\rho s}\beta_{0,s} dM_s^{0,a} + \int_0^{t\wedge\tau} e^{-\rho s}\lambda(\tilde{a}_s - a_s)ds \\
=& U_0 - \int_0^{t\wedge\tau} e^{-\rho s}\beta_{1,s} dM_s^{1,\tilde{a}} + \int_0^{t\wedge\tau} e^{-\rho s}\beta_{0,s} dM_s^{0,\tilde{a}} + \int_0^{t\wedge\tau} e^{-\rho s}\lambda(\tilde{a}_s - a_s)ds \\
&- \int_0^{t\wedge\tau} e^{-\rho s}\mu\alpha_s\beta_{1,s}(\tilde{a}_s - a_s)ds - \int_0^{t\wedge\tau} e^{-\rho s}\mu(1-\alpha_s)\beta_{0,s}(\tilde{a}_s - a_s)ds
\end{aligned}$$

Hence, $a_t = 0$ for all t is incentive compatible if and only if the drift term of the above expression is non-positive for any effort process $\tilde{a} \neq 0$; i.e.,

$$\lambda \leq \mu\alpha_t\beta_{1,t} + \mu(1-\alpha_t)\beta_{0,t}$$

for all t before termination. ■

7.2 Proofs in Section 3

Here we provide proofs for Property 2 and Theorem 3.1 here. Those of all the other properties are straightforward from the text and are therefore omitted.

7.2.1 Proof of Property 2

Proof. Note that the joint value function V must be nondecreasing in continuation value w. This is because in any region where V is strictly decreasing in w, the principal can benefit from paying out to the agent, contradicting the optimality of V. Let $A \subset \mathbb{R}_+$ denote the region of continuation values in which V is strictly increasing. Then the principal does not make any payment when $w \in A$ and $\mathbb{R}_+ \backslash A$ is the payout region. Since $\rho > r$, deferring payment becomes infinitely costly as $w \to +\infty$. Thus the payout region $\mathbb{R}_+ \backslash A$ is nonempty and there exists a $\bar{w} = \inf(\mathbb{R}_+ \backslash A)$.

By construction, V is strictly increasing for $w \in [0, \bar{w}]$ and is constant in a right neighborhood of $\bar{w}$, $(\bar{w}, \bar{w} + \Delta)$. Then, if $V'(\bar{w})$ exists, it must be zero. If $V'(\bar{w})$ does not exist, i.e., the left and the right derivatives are not equal, equation (9) is not defined at $w = \bar{w}$ and the coefficient in front of $V'(w)$ must be zero at $\bar{w}$. Notice that this coefficient is the drift of the continuation value. Hence, when $V'(\bar{w})$ does not exist, $\bar{w}$ is an absorbing payout boundary. As a result, no matter whether $V'(\bar{w})$ exists or not, the second and the third terms on the right-hand side of equation (9) must be zero when $w = \bar{w}$, leading to $V(\bar{w}) = \frac{z}{r} - \frac{\rho - r}{r}\bar{w}$.

By definition, the payout region is a subset of $(\bar{w}, +\infty)$. Actually, the payout region is $(\bar{w}, +\infty)$. Otherwise, there exists an interval $(\bar{w}' - \Delta, \bar{w}') \subset (\bar{w}, +\infty)$ such that V is strictly increasing on $[\bar{w}' - \Delta, \bar{w}']$ and is constant in a right neighborhood of $\bar{w}'$. It must be the case that $\bar{w}' < \infty$, since $\rho > r$ and deferring payment is infinitely costly as $w \to +\infty$. Then a similar argument regarding the existence of $V'(\bar{w})$ also applies here: no matter whether $V'(\bar{w}')$ exists or not, the second and the third terms on the right-hand side of equation (9) must be zero when $w = \bar{w}'$, and thus $V(\bar{w}') = \frac{z}{r} - \frac{\rho - r}{r}\bar{w}' < \frac{z}{r} - \frac{\rho - r}{r}\bar{w} = V(\bar{w})$, a contradiction to the non-decreasing property of V. Hence, the above defined $\bar{w}$ is the payout boundary and the payout region is $(\bar{w}, +\infty)$. As an immediate implication, the optimal payment is $dI^* = (w - \bar{w})^+$ and for $w \in [\bar{w}, +\infty)$, $V(w) = V(\bar{w})$.

The above proof has already shown that either $V'(\bar{w}) = 0$, or $V'(\bar{w})$ does not exist and $\rho\bar{w} - \mu(1 - \alpha^*(\bar{w}))\beta_0^*(\bar{w})$, the drift at $w = \bar{w}$, is 0.

The proof for $\bar{w} \leq \frac{\lambda}{\rho + \mu\bar{\alpha}}$ is straightforward from the text. ■

7.2.2 Proof of Theorem 3.1

Lemma 7.1 *For any $\bar{w} \in (0, \frac{\lambda}{\rho + \mu\bar{\alpha}}]$, let $\bar{V} \equiv \frac{z}{r} - \frac{\rho - r}{r}\bar{w}$. Then the ODE*

$$rV(w) = \max_{\alpha \in [0, \bar{\alpha}]} z - (\rho - r)w + \rho w V'(w) + (1 - \alpha)\mu[\bar{V} - V(w)] - (\lambda - \mu\alpha w)V'(w) \tag{19}$$

with boundary condition $V(0) = 0$ *has a unique solution on* $[0, \bar{w}]$.

Proof. For any $w < \frac{\lambda}{\rho+\bar{\alpha}\mu}$, since $\lambda - \mu\alpha w - \rho w > 0$, we can rearrange equation (19) to obtain

$$V' = \max_{\alpha\in[0,\bar{\alpha}]} \frac{z - (\rho - r)w + (1-\alpha)\mu[\bar{V} - V] - rV}{\lambda - \mu\alpha w - \rho w} .$$

Let

$$F(w, V) = \max_{\alpha\in[0,\bar{\alpha}]} \frac{z - (\rho - r)w + (1-\alpha)\mu[\bar{V} - V] - rV}{\lambda - \mu\alpha w - \rho w} .$$

For any fixed $\epsilon > 0$, for any $(w_1, V_1), (w_2, V_2) \in [0, \frac{\lambda}{\rho+\bar{\alpha}\mu} - \epsilon] \times [0, \bar{V}]$, there exists an M such that $|F(w_1, V_1) - F(w_2, V_2)| \leq M|V_1 - V_2|$. Then, by the Cauchy-Lipschitz theorem, the initial value problem has a unique solution over $[0, \frac{\lambda}{\rho+\bar{\alpha}\mu} - \epsilon]$. Further, notice that V is increasing and upper bounded, and therefore V does not explode as $w \to \bar{w}$. Then the maximum interval of existence reaches the boundary $\bar{w}$ for all $\bar{w} \leq \frac{\lambda}{\rho+\bar{\alpha}\mu}$. When $\bar{w} = \frac{\lambda}{\rho+\bar{\alpha}\mu}$, taking $\epsilon \to 0$, we can extend the solution over $\left[0, \frac{\lambda}{\rho+\bar{\alpha}\mu}\right]$. ■

Proposition 7.1 *Consider two ODEs*

$$rV_1 = max_{\alpha\in[0,\bar{\alpha}]} z - (\rho - r)w + \rho w V_1' + (1-\alpha)\mu[\bar{V}_1 - V_1] - (\lambda - \mu\alpha w)V_1'$$

and

$$rV_2 = max_{\alpha\in[0,\bar{\alpha}]} z - (\rho - r)w + \rho w V_2' + (1-\alpha)\mu[\bar{V}_2 - V_2] - (\lambda - \mu\alpha w)V_2' \ ,$$

where $\bar{V}_1 = \frac{z}{r} - \frac{\rho-r}{r}\bar{w}_1$, $\bar{V}_2 = \frac{z}{r} - \frac{\rho-r}{r}\bar{w}_2$, $\bar{w}_1 < \bar{w}_2 \leq \frac{\lambda}{\rho+\bar{\alpha}\mu}$*; and* $V_1(0) = V_2(0) = 0$*. Then* $V_1 > V_2$ *for* $w \in (0, \bar{w}_1)$.

Proof. Suppose the opposite holds. Note that $V_1'(0) > V_2'(0)$. Then, there exists a $w \in (0, \bar{w}_1)$ such that $V_1(w) = V_2(w)$. Define $\tilde{w} = \inf \{w \in (0, \bar{w}_1) : V_1(w) = V_2(w)\}$. By the continuity of V_1 and V_2, we have $V_1(\tilde{w}) = V_2(\tilde{w})$. Let α_2 be the α that solves the maximization problem for V_2 at $\tilde{w}$. Taking the difference between the two ODEs at $w = \tilde{w}$, we obtain

$$(\rho\tilde{w} + \mu\alpha_2\tilde{w} - \lambda) \cdot (V_1 - V_2)' + (1 - \alpha_2)\mu(\bar{V}_1 - \bar{V}_2) \le 0 \ .$$

Since $\alpha_2 < 1$ and $\bar{V}_1 - \bar{V}_2 > 0$,

$$(\rho\tilde{w} + \mu\alpha_2\tilde{w} - \lambda) \cdot (V_1 - V_2)' < 0 \ .$$

Since $\bar{w}_1 < \frac{\lambda}{\rho+\bar{\alpha}\mu}$, $\rho\tilde{w} + \mu\alpha_2\tilde{w} - \lambda < 0$. Thus, $V_1'(\tilde{w}) - V_2'(\tilde{w}) > 0$. Note that this inequality holds whenever $V_1 = V_2$. Since $V_1(w) - V_2(w)$ is continuous and the inequality is strict, it also holds for w close to $\tilde{w}$; i.e., $V_1'(w) - V_2'(w) > 0$ in $(\tilde{w} - \delta, \tilde{w})$ for some $\delta > 0$. By the definition of $\tilde{w}$, $V_1(w) - V_2(w) > 0$ for $w \in (\tilde{w} - \delta, \tilde{w})$. Then, it is impossible to have $V_1(\tilde{w}) = V_2(\tilde{w})$, a contradiction.
■

According to the above results, the candidate for the optimal payout boundary is the smallest $\bar{w} \in (0, \frac{\lambda}{\rho+\bar{\alpha}\mu}]$ such that the solution of ODE (15) satisfies $V(\bar{w}) = \frac{z}{r} - \frac{\rho-r}{r}\bar{w}$. The existence of such $\bar{w}$ is guaranteed by the continuity of V. Now we are ready to prove Theorem 3.1.

Proof. Here, we show that the contract that we derive is optimal among all contracts that always implement $a = 0$. Section 7.4 further establishes that such implementation is indeed optimal for the principal.

Let τ denote the first time that w_t hits zero. We first verify that the principal's value function can be induced by the proposed control processes in Property 5 and the proposed payment process $dI_t = (\beta_0 + w - \bar{w})^+ dY_t^0$. Note that by Property 3, $\beta_0 + w > \bar{w}$, so that $dI_t = (\beta_0 + w - \bar{w}) dY_t^0$. By Ito's Formula for jump processes,

$$e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) = B(w_0) + \int_0^{t\wedge\tau} e^{-rs}[(\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B'(w_s) - rB(w_s)]ds + \int_0^{t\wedge\tau} e^{-rs}[B(\bar{w}) - B(w_s)]dY_s^0 \ .$$

Under the optimal control processes, the HJB equation becomes

$$rB(w) = z + (\rho w - \beta_0\mu(1-\alpha))B^{'}(w) + (1-\alpha)\mu[B(\bar{w}) - B(w) - (w+\beta_0-\bar{w})] \ .$$

Thus,

$$\begin{aligned} B(w_0) = &\int_0^{t\wedge\tau} e^{-rs}[z + (1-\alpha_s)\mu(B(\bar{w}) - B(w_s) - (w_s + \beta_{0,s} - \bar{w}))]ds \\ &- \int_0^{t\wedge\tau} e^{-rs}[B(\bar{w}) - B(w_s)]dY_s^0 - e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) \ . \end{aligned}$$

Due to the fact that $Y_s^0 - (1-\alpha_s)\mu s$ is a martingale and $w_\tau = 0$, letting $t \to \infty$ and taking expectation on the right hand side of the above equation, we obtain

$$B(w_0) = \mathbb{E}(\int_0^{\tau} e^{-rs}[zds - (w_s + \beta_{0,s} - \bar{w})dY_s^0]) \ ,$$

which verifies that the principal's expected payoff given by equation (1) is indeed achieved with the proposed control and payment processes.

We then verify that the proposed contract is optimal. Since the cumulative payment process is increasing in time, without loss of generality, we write a general payment process as

$$I_t = I_t^c + I_t^d \ ,$$

where I_t^c is a continuous increasing process and I_t^d includes discrete upward jumps. By Ito's Formula for jump processes,

$$\begin{aligned} e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) = &B(w_0) + \int_0^{t\wedge\tau} e^{-rs}[(\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B^{'}(w_s) - rB(w_s)]ds \\ &- \int_0^{t\wedge\tau} e^{-rs}B^{'}(w_s)dI_s^c + \int_0^{t\wedge\tau} e^{-rs}[B(w_s + \beta_{0,s}) - B(w_s)]dY_s^0 \\ &+ \sum_{s\in[0,t\wedge\tau]} e^{-rs}[B(w_s + \beta_{0,s}\Delta Y_s^0 - \Delta I_s^d) - B(w_s + \beta_{0,s}\Delta Y_s^0)] \ , \end{aligned}$$

where $\Delta Y_s^0 \equiv Y_s^0 - Y_{s^-}^0$. We then rearrange the terms to get

$$
\begin{aligned}
B(w_0) =& e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) \\
&+ \int_0^{t\wedge\tau} e^{-rs}\{rB(w_s) - (\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B^{'}(w_s) - (1-\alpha_s)\mu[B(w+\beta_{0,s}) - B(w)]\}ds \\
&+ \int_0^{t\wedge\tau} B^{'}(w_s)e^{-rs}dI_s^c + \int_0^{t\wedge\tau}[B(w+\beta_{0,s}) - B(w)][(1-\alpha_s)\mu ds - dY_s^0] \\
&- \sum_{s\in[0,t\wedge\tau]} e^{-rs}[B(w_s + \beta_{0,s}\Delta Y_s^0 - \Delta I_t^d) - B(w_s + \beta_{0,s}\Delta Y_s^0)] \ .
\end{aligned}
$$

Taking expectation on both sides and using the fact that $Y_t^0 - \int_0^s (1-\alpha_s)\mu ds$ is a martingale, we obtain

$$
\begin{aligned}
B(w_0) =& \mathbb{E}(e^{-r(t\wedge\tau)}B(w_{t\wedge\tau})) \\
&+ \mathbb{E}\begin{pmatrix} \int_0^{t\wedge\tau} e^{-rs}\{rB(w_s) - (\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B^{'}(w_s) \\ -(1-\alpha_s)\mu[B(w+\beta_{0,s}) - B(w)]\}ds \end{pmatrix} \\
&+ \mathbb{E}(\int_0^{t\wedge\tau} B^{'}(w_s)e^{-rs}dI_s^c) \\
&- \mathbb{E}(\sum_{s\in[0,t\wedge\tau]} e^{-rs}[B(w_s + \beta_{0,s}\Delta Y_s^0 - \Delta I_t^d) - B(w_s + \beta_{0,s}\Delta Y_s^0)]) \ .
\end{aligned}
$$

Notice that

$$
rB(w) \geq z + (\rho w - \beta_0\mu(1-\alpha))B^{'}(w) + (1-\alpha)\mu[B(\bar{w}) - B(w) - (w + \beta_0 - \bar{w})]
$$

and for any incentive compatible contract,

$$
B(w+\beta_{0,s}) = B(\bar{w}) - (w + \beta_{0,s} - \bar{w}) \ .
$$

Moreover, since $B^{'}(w) \geq -1$,

$$
B(w_0) \geq \mathbb{E}(e^{-r(t\wedge\tau)}B(w_{t\wedge\tau})) + \mathbb{E}(\int_0^{t\wedge\tau} ze^{-rs}ds - \int_0^{t\wedge\tau} e^{-rs}dI_s^c) - \mathbb{E}(\sum_{s\in[0,t\wedge\tau]} e^{-rs}\Delta I_t^d) \ .
$$

Letting $t \to \infty$ and using the fact that $B(w)$ is bounded, we obtain

$$B(w_0) \geq \mathbb{E}(\int_0^\tau e^{-rs}(zds - dI_s)) \ .$$

Therefore, any function satisfying all these conjectured properties is indeed the value function for the principal. ■

7.3 Proofs in Section 4

7.3.1 Proof of Proposition 4.1

Proof. Recall from Property 2 that $\bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}}$. If $w = \bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, equation (18) is exactly equation (10), so $\alpha^*(w) = \bar{\alpha}$.

For $w \in \left(0, \frac{\lambda}{\rho+\mu\bar{\alpha}}\right)$, equation (15) is equivalent to

$$V'(w) = \max_{\alpha\in[0,\bar{\alpha}]} \frac{z - (\rho - r)w + (1-\alpha)\mu[V(\bar{w}) - V(w)] - rV(w)}{\lambda - \rho w - \mu\alpha w}. \tag{20}$$

Let $G(\alpha; w) = \frac{z-(\rho-r)w+(1-\alpha)\mu[V(\bar{w})-V(w)]-rV(w)}{\lambda-\rho w-\mu\alpha w}$, which is obviously continuous in both α and w. Property 5 establishes that the maximizer of the right-hand side (RHS) of equation (20) must be 0 or $\bar{\alpha}$. So to figure out $\alpha^*(w)$, it suffices to compare $G(0; w)$ with $G(\bar{\alpha}; w)$, taking as given $V(0) = 0$ and $V(\bar{w})$.

For $w \in \left(0, \frac{\lambda}{\rho+\mu\bar{\alpha}}\right)$, $G(\bar{\alpha}; w) \geq G(0; w)$ is equivalent to

$$w[z - (\rho - r)w - rV] \geq [\lambda - (\mu + \rho)w](V(\bar{w}) - V(w)) \ . \tag{21}$$

Let $\hat{w}_0 = \min\left\{\bar{w}, \frac{\lambda}{\rho+\mu}\right\}$. For any $w \in (0, \hat{w}_0)$, the left-hand side of inequality (21) is negative, but its right-hand side is positive. So it fails, establishing the optimality of $\alpha(w) = 0$ in this range.

Now we establish the optimality of $\alpha(w) = \bar{\alpha}$ for w in the vicinity of $\bar{w}$. Note that by equation (10), inequality (21) is equivalent to

$$w(\rho - r)(\bar{w} - w) \geq [\lambda - (\mu + \rho + r)w](V(\bar{w}) - V(w)), \tag{22}$$

which holds for all $w \geq \frac{\lambda}{\rho+\mu+r}$. So if $\bar{w} \in (\frac{\lambda}{\rho+\mu+r}, \frac{\lambda}{\rho+\mu\bar{\alpha}}]$, $\alpha^*(w) = \bar{\alpha}$ for all $w \in (\frac{\lambda}{\rho+\mu+r}, \bar{w}]$.

Note that inequality (22) is equivalent to $\frac{\bar{w}-w}{V(\bar{w})-V(w)} \geq \frac{\lambda-(\mu+\rho+r)w}{(\rho-r)w}$. If $\bar{w} \leq \frac{\lambda}{\rho+\mu+r} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$, by Lemma 4.1 (whose proof does not require Proposition 4.1), $\bar{w}$ is reflective so that $V'(\bar{w}) = 0$. Then by L'Hôpital's rule, $\lim_{w\to\bar{w}^-} \frac{\bar{w}-w}{V(\bar{w})-V(w)} = \lim_{w\to\bar{w}^-} \frac{1}{V'(w)} = +\infty$, while $\lim_{w\to\bar{w}^-} \frac{\lambda-(\mu+\rho+r)w}{w(\rho-r)} = \frac{\lambda-(\mu+\rho+r)\bar{w}}{(\rho-r)\bar{w}} < +\infty$. Hence, there also exists a $\hat{w}_{\bar{\alpha}} < \bar{w}$, such that $\alpha(w) = \bar{\alpha}$ for all $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$.

$\beta_0^*(w)$ for $w \in (0, \hat{w}_0) \cup (\hat{w}_{\bar{\alpha}}, \bar{w}]$ results from equation (14). ■

Here we provide the closed-form solutions to equations (17) and 18). As a first-order linear ODE, equation (17) has general solutions

$$V(w) = \frac{\rho-r}{r+\mu-\rho}(\frac{\lambda}{\rho}-w) + \frac{\mu V(\bar{w}) + z - (\rho-r)\frac{\lambda}{\rho}}{r+\mu} + K(\frac{\lambda}{\rho}-w)^{\frac{r+\mu}{\rho}}, \quad (23)$$

which are all strictly concave in $(0, \bar{w})$. From $V(0) = 0$, we can pin down for $w \in (0, \hat{w}_0)$ that $K = -\frac{\rho(\rho-r)}{(r+\mu)(r+\mu-\rho)} \cdot \frac{\lambda}{\rho}^{-\frac{r+\mu-\rho}{\rho}} - \frac{\mu V(\bar{w})+z}{r+\mu} \cdot \frac{\lambda}{\rho}^{-\frac{r+\mu}{\rho}}$.

Also as a first-order linear ODE, equation (18) has general solutions

$$\begin{aligned} V(w) &= \frac{\rho-r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)}(\frac{\lambda}{\rho+\mu\bar{\alpha}}-w) + \frac{(1-\bar{\alpha})\mu V(\bar{w}) + z - (\rho-r)\frac{\lambda}{\rho+\bar{\alpha}\mu}}{r+\mu(1-\bar{\alpha})} \\ &\quad + K(\frac{\lambda}{\rho+\mu\bar{\alpha}}-w)^{\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}}} \end{aligned} \quad (24)$$

if $r + (1-\bar{\alpha})\mu \neq \rho + \bar{\alpha}\mu$, and

$$\begin{aligned} V(w) &= -\frac{\rho-r}{\rho+\mu\bar{\alpha}}(\frac{\lambda}{\rho+\mu\bar{\alpha}}-w)\ln(\frac{\lambda}{\rho+\mu\bar{\alpha}}-w) + \frac{(1-\bar{\alpha})\mu V(\bar{w}) + z - (\rho-r)\frac{\lambda}{\rho+\bar{\alpha}\mu}}{r+\mu(1-\bar{\alpha})} \\ &\quad + K(\frac{\lambda}{\rho+\mu\bar{\alpha}}-w) \end{aligned} \quad (25)$$

if $r + (1-\bar{\alpha})\mu = \rho + \bar{\alpha}\mu$. It is shown later in the proof of Proposition 4.2 that the solutions that are increasing in $(0, \bar{w})$ are strictly convex in $(0, \bar{w})$ if $r + (1-\bar{\alpha})\mu < \rho + \bar{\alpha}\mu$ and $K < 0$, linear if $r + (1-\bar{\alpha})\mu < \rho + \bar{\alpha}\mu$ and $K = 0$, and strictly concave in $(0, \bar{w})$ otherwise.

With the closed-form solutions and their concavity properties discussed above, we show the following proposition:

Proposition 7.2 *If* $\bar{w} \geq \frac{\lambda}{\rho+\mu+r}$*, then* $\hat{w}_0 = \hat{w}_{\bar{\alpha}}$*.*

To prove Proposition 7.2, we first prove Lemma 7.2, which articulates that the optimal α takes values in $\{0, \overline{\alpha}\}$ almost surely.

Lemma 7.2 *There does not exist an interval* (w_1, w_2) *such that* $w \cdot V'(w) = V(\bar{w}) - V(w)$ *for all* $w \in (w_1, w_2)$*.*

Proof. Suppose the contrary. Then $w \cdot V'(w) = V(\bar{w}) - V(w)$ implies

$$V(w) = \frac{c}{w} + V(\bar{w}) \tag{26}$$

in (w_1, w_2) for some constant c. Plugging $w \cdot V'(w) = V(\bar{w}) - V(w)$ into the HJB equation (9) we obtain

$$V(w) = \frac{z - (\rho - r)w + (\rho + \mu - \lambda/w)V(\bar{w})}{r + \rho + \mu - \lambda/w}. \tag{27}$$

It is straightforward to verify that equations (26) and (27) cannot both be satisfied in any interval. ■

Lemma 7.3 shows that the convexity of V in an interval below the payout boundary $\bar{w}$ is "contagion" up to $\bar{w}$.

Lemma 7.3 *If there exists an interval* $[w_1, w_2) \subset (0, \bar{w})$ *such that* $w_1 \cdot V'(w_1) \geq V(\bar{w}) - V(w_1)$ *and* V *is convex in* (w_1, w_2)*, then* $\alpha^*(w) = \bar{\alpha}$ *for all* $w \in (w_1, \bar{w}]$ *and* V *is convex in* $[w_1, \bar{w}]$*.*

Proof. If V is convex in (w_1, w_2), since V is continuously differentiable in $(0, \bar{w})$, $w \cdot V'(w) + V(w)$ is strictly increasing in $[w_1, w_2)$. Given that $w_1 \cdot V'(w_1) \geq V(\bar{w}) - V(w_1)$, we have $w \cdot V'(w) > V(\bar{w}) - V(w)$ for all $w \in (w_1, w_2)$. So there exists $w_3 \in (w_2, \bar{w})$ such that $w \cdot V'(w) > V(\bar{w}) - V(w)$ for all $w \in (w_1, w_3)$. Iteration of this argument yields $w \cdot V'(w) > V(\bar{w}) - V(w)$

and thus $\alpha^*(w) = \bar{\alpha}$ for all $w \in (w_1, \bar{w})$. By Proposition 4.1, $\alpha^*(\bar{w}) = \bar{\alpha}$ as well.

Given that $\alpha^*(w) = \bar{\alpha}$ for all $w \in (w_1, \bar{w}]$, the specific solution to equation (18) that matches the value function V in $[w_1, w_2)$ must also match V in $[w_1, \bar{w}]$. Since V is convex in $[w_1, w_2)$, that specific solution must be given by equation (24) with $K \leq 0$ and $r+(1-\bar{\alpha})\mu < \rho+\bar{\alpha}\mu$. This proves the convexity of V in $[w_1, \bar{w}]$. ■

With Lemmas 7.2 and 7.3, we can now prove Proposition 7.2.

Proof. Let $\hat{W} \equiv \left\{w \in (0, \bar{w}) : w \cdot V'(w) = V(\bar{w}) - V(w)\right\}$. We are to show that $\hat{W}$ is a singleton if $\bar{w} \geq \frac{\lambda}{\rho+\mu+r}$. By Proposition 4.1, $\hat{W}$ is non-empty and has a maximum. Without loss of generality, assume $\hat{w}_{\bar{\alpha}} = \max \hat{W}$. Then V must be strictly concave in $(0, \hat{w}_{\bar{\alpha}}]$. To see this, Lemma 7.2 and the properties of the general solutions to equations (17) and 18) imply that V must be piecewise concave or convex in $(0, \hat{w}_{\bar{\alpha}}]$. If there is an interval $(w_1, w_2) \subset (0, \hat{w}_{\bar{\alpha}}]$ such that V is convex in it, then by Lemma 7.3, $\alpha^*(w) = \bar{\alpha}$ for all $w \in (w_1, \bar{w}]$, contradicting the fact that $\hat{w}_{\bar{\alpha}} = \max \hat{W}$.

Note that equation (27) holds for $w = \hat{w}_{\bar{\alpha}}$. Plug it into $V'(\hat{w}_{\bar{\alpha}}) = \frac{V(\bar{w})-V(\hat{w}_{\bar{\alpha}})}{\hat{w}_{\bar{\alpha}}}$, we have $V'(\hat{w}_{\bar{\alpha}}) = \frac{\rho-r}{r+\rho+\mu}(1 - \frac{\frac{\lambda}{r+\mu+\rho}-\bar{w}}{\frac{\lambda}{r+\mu+\rho}-\hat{w}_{\bar{\alpha}}})$. Similarly, if there exists $\hat{w}' \in \hat{W}$ such that $\hat{w}' < \hat{w}_{\bar{\alpha}}$, then $V'(\hat{w}') = \frac{\rho-r}{r+\rho+\mu}(1 - \frac{\frac{\lambda}{r+\mu+\rho}-\bar{w}}{\frac{\lambda}{r+\mu+\rho}-\hat{w}'})$. If $\bar{w} \geq \frac{\lambda}{\rho+\mu+r}$, then we have $V'(\hat{w}') \leq V'(\hat{w}_{\bar{\alpha}})$, contradicting the concavity of V in $(0, \hat{w}_{\bar{\alpha}}]$. ■

7.3.2 Proof of Proposition 4.2

Proof. By Proposition 4.1, $\alpha^*(w) = \bar{\alpha}$ if $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$, so here we focus on the solutions to equation (18) when studying the property of the payout boundary $\bar{w}$. Let V_K be the solution with constant K in equation (24) or (25).

Case 1: If $r + (1-\bar{\alpha})\mu > \rho + \bar{\alpha}\mu$, we must have $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$, and thus $\bar{w}$ is reflective by Lemma 4.1. To see this, equation (24) yields

$$V_K' = -\frac{\rho - r}{r + (1-\bar{\alpha})\mu - (\rho + \bar{\alpha}\mu)} - K\frac{r + (1-\bar{\alpha})\mu}{\rho + \mu\bar{\alpha}}\left(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w\right)^{\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}}-1}. \tag{28}$$

Since the first term of the right-hand side of equation (28) is negative, K must be negative, otherwise $V_K^{'} < 0$ for all $w \leq \frac{\lambda}{\rho+\mu\bar{\alpha}}$, contradicting the optimality of $\alpha^*(w) = \bar{\alpha}$ for $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$. Since $\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}} - 1 > 0$, V_K is concave. Moreover, as $w \to \frac{\lambda}{\rho+\mu\bar{\alpha}}$, $V_K^{'} \to -\frac{\rho-r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)} < 0$, contradicting the optimality of $\alpha^*(w) = \bar{\alpha}$ for $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$ if $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$.

Case 2: If $r + (1-\bar{\alpha})\mu = \rho + \bar{\alpha}\mu$, we have

$$V_K^{'} = -K + \frac{\rho - r}{\rho + \mu\bar{\alpha}} + \frac{\rho - r}{\rho + \mu\bar{\alpha}} \ln(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w) \ .$$

Regardless of the value of K, V_K is concave and as $w \to \frac{\lambda}{\rho+\mu\bar{\alpha}}$, $V_K^{'} \to -\infty$. Thus, analogous to the previous case, it must be that $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$ and $\bar{w}$ is reflective.

Case 3: If $r+(1-\bar{\alpha})\mu < \rho+\bar{\alpha}\mu$, since $-\frac{\rho-r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)} > 0$, K in equation (24) can be either positive or negative. Equation (28) yields

$$V_K^{''} = K\frac{r + (1-\bar{\alpha})\mu}{\rho + \mu\bar{\alpha}}(\frac{r + (1-\bar{\alpha})\mu}{\rho + \mu\bar{\alpha}} - 1)(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w)^{\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}} - 2} \ .$$

If $K > 0$, since $\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}} - 1 < 0$, $V_K^{''} < 0$ so that V_K is concave. Moreover, as $w \to \frac{\lambda}{\rho+\mu\bar{\alpha}}$, $V_{\bar{\alpha}}^{'} \to -\infty$. Again, it must be that $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$, and $\bar{w}$ is reflective.

If $K = 0$, then $V_K^{'}(w) = -\frac{\rho-r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)} > 0$ for all $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$. Thus we must have $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ as an absorbing state.

If $K < 0$, $V_K^{''} > 0$ so that V_K is strictly convex. Thus, the value function V satisfies $V^{'} > 0$ for all $w < \frac{\lambda}{\rho+\mu\bar{\alpha}}$. This implies that $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, and $\bar{w}$ is absorbing by Lemma 4.1.

To summarize all the cases above, we have $\bar{w}$ is absorbing (i.e., $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$) if and only if $r+(1-\bar{\alpha})\mu < \rho+\bar{\alpha}\mu$ and $K \leq 0$; i.e., if and only if V is (weakly) convex in $(\hat{w}_{\bar{\alpha}}, \bar{w})$.

For the "if" claim, it is sufficient to show that when $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$ and z is large enough, it must be the case that $K \leq 0$. Suppose the contrary that K is strictly positive for an arbitrarily large z. Let $\hat{w}$ denote the smallest w at which the principal switches from $\alpha = 0$ to $\alpha = \bar{\alpha}$. Then equation (24) implies

$$V(\hat{w}) \leq \frac{\rho - r}{r + (1-\bar{\alpha})\mu - (\rho + \bar{\alpha}\mu)} \frac{\lambda}{\rho + \mu\bar{\alpha}} + \frac{(1-\bar{\alpha})\mu\bar{V} + z - (\rho - r)\frac{\lambda}{\rho+\bar{\alpha}\mu}}{r + \mu(1-\bar{\alpha})} . \tag{29}$$

Further notice that from the solution to the ODE with $\alpha = 0$ control,

$$V(\hat{w}) = [1-(1-\frac{\rho}{\lambda}\hat{w})^{\frac{r+\mu}{\rho}}][\frac{\rho - r}{(r+\mu)(r+\mu-\rho)}\lambda + \frac{\mu\bar{V} + z}{r+\mu}] - \frac{\rho - r}{r+\mu-\rho}\hat{w} . \tag{30}$$

Since $\hat{w} < \bar{w} \leq \frac{\lambda}{\rho+\bar{\alpha}\mu}$, the coefficient in front of $\frac{z}{r}$ on the right-hand side of equation (30) is smaller then 1.[13] On the other hand, the coefficient in front of $\frac{z}{r}$ on the right-hand side of inequality (29) is 1. Thus, when z is large enough, inequality (29) cannot be satisfied, which is a contradiction. As a result, when z is sufficently large, V is (weakly) convex in $(\hat{w}_{\bar{\alpha}}, \bar{w})$ and $\bar{w}$ is absorbing.

Now we prove the "only if" claim; i.e., $\bar{w}$ is absorbing only if $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$ and $z/\lambda \geq \theta(r, \rho, \mu, \bar{\alpha})$, where $\theta(r, \rho, \mu, \bar{\alpha})$ is defined by (34).

From equation (23), we have

$$V^{'}(\hat{w}) = \frac{\rho - r}{r+\mu-\rho}(1 - \frac{\rho}{\lambda}\hat{w})^{\frac{r+\mu}{\rho}-1} + \frac{\mu V(\bar{w}) + z}{\lambda}(1 - \frac{\rho}{\lambda}\hat{w})^{\frac{r+\mu}{\rho}-1} - \frac{\rho - r}{r+\mu-\rho} . \tag{31}$$

On the other hand, by Property 5, we have $V'(\hat{w}) = \frac{V(\bar{w}) - V}{\hat{w}}$. Plugging this into equation (17), we have

$$V^{'}(\hat{w}) = \frac{\rho - r}{r+\rho+\mu}(1 - \frac{\frac{\lambda}{r+\mu+\rho} - \bar{w}}{\frac{\lambda}{r+\mu+\rho} - \hat{w}}) . \tag{32}$$

As $\hat{w}$ increases from 0 to $\frac{\lambda}{r+\mu+\rho}$, the right-hand side of equation (31) is decreasing from $\frac{\mu V(\bar{w})+z}{\lambda}$, and that of equation (32) is increasing from $\frac{z - V(\bar{w})}{\lambda}$ to $+\infty$. Thus, there exists a unique $\hat{w} \in (0, \frac{\lambda}{r+\mu+\rho})$ such that both equations hold simultaneously.

Next, we show that V_K is convex if and only if $\hat{w} \geq \frac{\lambda}{2(\rho+\mu\bar{\alpha})}$. Observe that

[13] Note that $\bar{V} = \frac{z}{r} - \frac{\rho - r}{r}\bar{w}$.

$V(\hat{w})$ should also satisfy equation (24), and thus

$$V'(\hat{w}) = -\frac{\rho - r}{r + (1-\bar{\alpha})\mu - (\rho + \bar{\alpha}\mu)} - K\frac{r + (1-\bar{\alpha})\mu}{\rho + \mu\bar{\alpha}}\left(\frac{\lambda}{\rho + \mu\bar{\alpha}} - \hat{w}\right)^{\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}} - 1} . \tag{33}$$

We have shown that V_K is convex if and only if $r + (1 - \bar{\alpha})\mu < \rho + \bar{\alpha}\mu$ and $K \leq 0$. From equations (32) and (33),

$$K \leq 0 \implies \frac{\rho - r}{r + \rho + \mu}\left(1 - \frac{\frac{\lambda}{r+\mu+\rho} - \bar{w}}{\frac{\lambda}{r+\mu+\rho} - \hat{w}}\right) \geq -\frac{\rho - r}{r + (1-\bar{\alpha})\mu - (\rho + \bar{\alpha}\mu)},$$

where $\bar{w} = \frac{\lambda}{\rho + \bar{\alpha}\mu}$. This reduces to $\hat{w} \geq \frac{\lambda}{2(\rho+\mu\bar{\alpha})}$. Notice that if $r + (1 - \bar{\alpha})\mu < \rho + \bar{\alpha}\mu$, $\frac{\lambda}{2(\rho+\mu\bar{\alpha})} < \frac{\lambda}{r+\mu+\rho}$.

Therefore, V_K is convex only if $r + (1 - \bar{\alpha})\mu < \rho + \bar{\alpha}\mu$ (i.e., $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$) and the right-hand sides of equations (31) and (32) intersect at some $\hat{w} \in [\frac{\lambda}{2(\rho+\mu\bar{\alpha})}, \frac{\lambda}{r+\mu+\rho})$. The second condition holds if and only if

$$\left(\frac{\rho - r}{r + \mu - \rho} + \frac{\mu\bar{V} + z}{\lambda}\right)\left(1 - \frac{\rho}{\lambda}\cdot\frac{\lambda}{2(\rho + \mu\bar{\alpha})}\right)^{\frac{r+\mu}{\rho} - 1} - \frac{\rho - r}{r + \mu - \rho} \geq \frac{\rho - r}{r + \rho + \mu}\left(1 - \frac{\frac{\lambda}{r+\mu+\rho} - \frac{\lambda}{\rho+\mu\bar{\alpha}}}{\frac{\lambda}{r+\mu+\rho} - \frac{\lambda}{2(\rho+\mu\bar{\alpha})}}\right),$$

which is equivalent to

$$\begin{aligned} \frac{z}{\lambda} &\geq \frac{r(\rho - r)}{\mu + r}\left\{\frac{2(\rho + \mu\bar{\alpha})}{\rho + 2\mu\bar{\alpha}}\frac{2\mu\bar{\alpha}}{(r + \mu - \rho)[(\rho + \mu\bar{\alpha}) - (r + \mu(1-\bar{\alpha}))]} - \frac{1}{r + \mu - \rho} + \frac{\mu}{\rho + \mu\bar{\alpha}}\right\} \\ &\equiv \theta\left(r, \rho, \mu, \bar{\alpha}\right) . \end{aligned} \tag{34}$$

Finally, if $\bar{w}$ is absorbing, $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}} > \frac{\lambda}{\rho+\mu+r}$, so we have $\hat{w}_0 = \hat{w}_{\bar{\alpha}}$ by Proposition 7.2. (We write $\hat{w} = \hat{w}_0 = \hat{w}_{\bar{\alpha}}$ in this case.) ■

7.4 The Optimality of No Shirking

In the proof of Theorem 3.1 in Section 7.2, we show that among contracts that always implement $a = 0$, the contract that we derive is optimal. This subsection further establishes the optimality of such implementation.

First, if $z > \lambda$, it is suboptimal for the principal to implement $a > 0$ in the payout region, $[\bar{w}, +\infty)$. To see this, consider any contract that implements $a_t > 0$ for some $w_t > \bar{w}$. This implies that in $[t, t+dt]$, the agent receives a private benefit of $\lambda a_t dt$. Instead, the principal could implement $a_t = 0$ (which generates additional synergy $z \cdot a_t dt$), and increases the payment to the agent by $\lambda a_t dt$ in $[t, t+dt]$, without altering the contract afterwards. This raises the principal's payoff by $(z - \lambda)\, a_t dt > 0$, while leaving the dynamics of the agent continuation value unchanged. Iteration of this argument rules out profitable deviation from $a = 0$ in the payout region. Thus, we only need to consider the possibility of shirking in the *no-payment region* henceforth. We will rule out the profitability of deviation of $a = 1$ and $a \in (0,1)$, respectively.

7.4.1 Shirking with $a = 1$

If the principal implements $a = 1$ for some $w_t \in (0, \bar{w})$, then the agent's IC constraint is $\mu\alpha\beta_1 + \mu(1-\alpha)\beta_0 \leq \lambda$, and her continuation value follows

$$dw_t = \rho w_t dt - \lambda dt - \beta_{1,t}[dY_{1,t} - \mu\alpha_t dt] \; .$$

Thus, to rule out the profitability of such deviation, we need to show that for any $w \in (0, \bar{w})$,

$$rV(w) \geq \max_{\alpha,\beta_1} \lambda + [\rho w - \lambda + \alpha\mu\beta_1]V'(w) + \alpha\mu[V(w-\beta_1) - V(w)] - (\rho - r)w \; . \quad (35)$$

If $\beta_1 \geq 0$, then

$$\begin{aligned} \text{RHS of inequality (35)} \;&\leq\; \max_{\alpha} z + (\rho w - \lambda + \alpha\mu w)\, V'(w) - (\rho - r)w \\ &<\; \max_{\alpha} z + (\rho w - \lambda + \alpha\mu w)\, V'(w) + \mu\,(1-\alpha)\,[V(\bar{w}) - V(w)] - (\rho - r)w \\ &\leq\; rV\,(w) \;\; , \end{aligned}$$

where the first two inequalities follow the fact that V is increasing in w, and the third inequality holds because its LHS is the flow value achieved with $\beta_1 = w$ when implementing $a = 0$ and thus the LHS is dominated by the optimal flow value rV. Hence, it suffices to show inequality (35) holds for $\beta_1 < 0$.

Denote the objective function of the RHS of inequality (35) by D. If $\beta_1 \leq 0$, we show that D can achieve its maximum only when $\beta_1 = 0$ or $\beta_1 = -(\bar{w} - w)$. In particular, for $\beta_1 < -(\bar{w} - w)$, we have $V(w - \beta_1) = V(\bar{w})$ and $\partial D / \partial \beta_1 = \alpha \mu V'(w) > 0$. Thus D can achieve its maximum only when $\beta_1 \in [-(\bar{w} - w), 0]$. For $\beta_1 > -(\bar{w} - w)$, $\partial D / \partial \beta_1 = \alpha \mu \left[V'(w) - V'(w - \beta_1)\right]$. If $V'(w) \geq V'(w - \beta_1)$ for all $\beta_1 \in [-(\bar{w} - w), 0]$, then $\partial D / \partial \beta_1 > 0$ and D is maximized with $\beta_1 = 0$. If $V'(w) < V'(w - \beta_1)$ for some $\beta_1 \in [-(\bar{w} - w), 0]$, then V is not globally concave. By Proposition 4.2, V is concave in $(0, \hat{w}_0)$ and convex in $[\hat{w}_0, \bar{w}]$. This implies the existence of $\hat{\beta}$ such that $w - \hat{\beta} \in [\hat{w}_0, \bar{w}]$ and that $\partial D / \partial \beta_1 = \alpha \mu \left[V'(w) - V'(w - \beta_1)\right] < 0$ if and only if $\beta_1 < \hat{\beta}$. Therefore, if D is maximized with $\beta_1 \in [-(\bar{w} - w), \hat{\beta})$, then the maximizer is $\beta_1 = -(\bar{w} - w)$. If D is instead maximized with $\beta_1 \in \left[\hat{\beta}, 0\right]$, then the maximizer is $\beta_1 = 0$. Since we have already shown that inequality (35) holds for $\beta_1 \geq 0$, we only needs to show that it holds for $\beta_1 = -(\bar{w} - w)$.

Note also that D is linear in α, so D is maximized with either $\alpha = 0$ or $\alpha = \bar{\alpha}$.[14] If D is maximized with $\beta_1 = -(\bar{w} - w)$ and $\alpha = 0$, then the RHS of inequality (35) equals

$$\lambda + (\rho w - \lambda) V'(w) - (\rho - r) w \ ;$$

If D is maximized with $\beta_1 = -(\bar{w} - w)$ and $\alpha = \bar{\alpha}$, then the RHS of inequality (35) equals

$$\lambda + (\rho w - \lambda) V'(w) + \bar{\alpha} \mu [V(\bar{w}) - V(w) - (\bar{w} - w) V'(w)] - (\rho - r) w \, .$$

For both cases, we have

$$\text{RHS of inequality (35)} < z + (\rho w - \lambda) V'(w) + \mu [V(\bar{w}) - V(w)] - (\rho - r) w \leq r V(w) \, ,$$

where the last inequality results from equation (15). Therefore, deviation to $a = 1$ is never profitable.

[14] If D is maximized at some interior value of α, the coefficient of α must be zero and thus it is equivalent to evaluating D at $\alpha = 0$.

7.4.2 Shirking with $a \in (0,1)$

If the principal instead implements $a \in (0,1)$ for some $w_t \in (0, \bar{w})$, then the agent's continuation value follows

$$dw_t = \rho w_t dt - a\lambda dt + \beta_{0,t}\left[dY_{0,t} - \mu\left(1-\alpha_t\right)(1-a)dt\right] - \beta_{1,t}[dY_{1,t} - \mu\alpha_t a dt] \; .$$

To guarantee that the agent does not choose either $a = 0$ or $a = 1$, her IC constraint is

$$\mu\alpha\beta_1 + \mu(1-\alpha)\beta_0 = \lambda.$$

To rule out the profitability of such deviation, we need to show that for any $w \in (0, \bar{w})$,

$$\begin{aligned} rV(w) \geq \max_{\alpha,\beta_0,\beta_1} & z(1-a) + a\lambda + [\rho w - a\lambda - \beta_0\mu(1-\alpha)(1-a) + \mu\alpha\beta_1 a]V'(w) \\ & + (1-\alpha)\mu(1-a)[V(\bar{w}) - V(w)] + \alpha\mu a[V(w-\beta_1) - V(w)] - (\rho - r)w \; . \end{aligned} \tag{36}$$

From the IC condition, the RHS of inequality (36) equals

$$\begin{aligned} & \max_{\alpha,\beta_0,\beta_1} z(1-a) + a\lambda + [\rho w - \beta_0\mu(1-\alpha)]V'(w) \\ & \quad + (1-\alpha)\mu(1-a)[V(\bar{w}) - V(w)] + \alpha\mu a[V(w-\beta_1) - V(w)] - (\rho - r)w \; . \end{aligned}$$

If $\beta_1 \geq 0$, then equation (15) implies that inequality (36) holds. If $\beta_1 < 0$, then due to the IC constraint, the RHS of inequality (36) equals

$$\begin{aligned} & \max_{\alpha,\beta_1} z(1-a) + a\lambda + [\rho w - \lambda]V'(w) + (1-\alpha)\mu(1-a)[V(\bar{w}) - V(w)] \\ & \quad + \alpha\mu a[V(w-\beta_1) - V(w)] + \mu\alpha\beta_1 V'(w) - (\rho - r)w \\ & \quad < \max_{\alpha} z + [\rho w - \lambda]V'(w) + [(1-\alpha)(1-a) + a\alpha]\mu[V(\bar{w}) - V(w)] - (\rho - r)w \\ & \quad < z + [\rho w - \lambda]V'(w) + \mu[V(\bar{w}) - V(w)] - (\rho - r)w \leq rV(w) \, , \end{aligned}$$

where the last inequality again results from equation (15). Thus, deviation to

$a \in (0, 1)$ is never profitable either.