CIRJE-F- 1182

Principal-Agent Problems with Hidden Savings in Continuous Time: Validity of the First-Order Approach

Tomoyuki Nakajima
The University of Tokyo

December 2021

Principal-agent problems with hidden savings in continuous time: Validity of the first-order approach

Tomoyuki Nakajima*

December 21, 2021

Abstract

In this paper, I establish the validity of the first-order approach to the continuous-time principal-agent problem with hidden savings. The agent's problem, which is non-Markovian, is formulated using a stochastic HJB equation. Without loss of generality, the payment process is designed so that it is optimal for the agent to choose zero savings. Then, the principal's problem can be expressed as maximizing her expected profit subject to two SDEs: one equation describing the agent's continuation utility process, and the other being the Euler equation concerning the agent's marginal utility process.

Keywords: moral hazard; hidden savings; continuous time; weak formulation; first-order approach; stochastic HJB equations.
JEL Classification numbers: C61; D81; D82; D86; E21.

*Faculty of Economics, University of Tokyo, and the Canon Institute for Global Studies. Email: tomoyuki.nakajima@gmail.com. Web page: `https://sites.google.com/view/tomoyukinakajima/home`. I thank Masaaki Fujii for helpful discussions related to this project. Financial support from the JSPS Grant-in-Aid for Scientific Research (21K01385, 18H03638) is gratefully acknowledged.

1 Introduction

In this paper, I consider a dynamic principal-agent problem in continuous time, in which the agent can save/borrow without being observed by the principal. The main result is the validity of the first-order approach, in the sense that the principal's problem is formulated as maximizing her expected profit subject to stochastic differential equations (SDE's)–one equation describing the evolution of the agent's continuation utility, and the other corresponding to the agent's Euler equation for his utility maximization problem.

Specifically, I study two classes of dynamic principal-agent problems. In the first one, the agent supplies unobserved effort to produce output. Holmstrom and Milgrom (1987) is the seminal work that analyzes such a problem in continuous time. Their analysis is restricted to the case where the agent has an exponential utility function. Sannikov (2008) extends their framework to a more general setting.

In the second model, the agent does not provide effort. Instead, he manages an asset for the principal, whose return is only observed by the agent. This problem is considered in the continuous-time framework by DeMarzo and Sannikov (2006), and Biais et al. (2007).

In both models, I study the optimal contracting problem of the principal when the agent has a hidden access to risk-free borrowing and lending. In the existing literature, two papers have addressed a closely related question. Williams (2015) considers a hidden effort model as Sannikov (2008) but allows for hidden savings. As in Holmstrom and Milgrom (1987), Williams (2015) assumes the exponential utility function for the agent. In that framework, he considers a relaxed problem in which the principal maximizes her expected profit subject to the first-order condition (Euler equation) of the agent, without establishing its sufficiency for the agent's problem. It is then verified that the solution to the relaxed problem is indeed incentive compatible for the agent.

Di Tella and Sannikov (2021) consider the asset management problem as in DeMarzo and Sannikov (2006), and Biais et al. (2007). They also allow for hidden savings for the agent and restrict attention to the case where the agent's utility function is of the CRRA form. Then they establish the first-order approach is valid under some additional condition for the contract offered by the principal, which is shown to be satisfied at optimum.

In this paper, for both types of the problem, I establish the validity of the first-order approach without imposing any parametric assumption on the utility function of the agent (aside from the standard assumptions such as monotonicity, concavity, differentiability, and integrability). I express the agent's utility maximization problem as a non-Markovian dynamic programming problem, and then derives the optimality condition as a stochastic Hamilton-Jacobi-Bellman (HJB) equation. The stochastic HJB equation is first studied by Peng (1992), and my argument is based on the results in Øksendal and Sulem (2019). When considering the principal's problem, without loss of generality, I restrict attention to those contracts for which the agent chooses zero savings. For such contracts, the agent's incentive compatibility conditions are summarized by two SDE's: one for the continuation utility, and the other for the Euler equation, which establishes the validity of the first-order approach.

The rest of the paper is organized as follows. Section 2 considers the model with hidden effort. Section 3 studies the model with hidden returns. Section 4 concludes.

2 Hidden effort

In this section, I consider a version of the dynamic principal-agent model of Sannikov (2008), modified so that the agent can save/borrow, hidden from the principal.

2.1 The model

Time is continuous and indexed by $t \in [0, T]$, where $0 < T$. The time horizon can be made infinite by considering the limit $T \to +\infty$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, on which a standard Brownian motion $B : [0, T] \times \Omega \to \mathbb{R}$ is defined. Let $\mathbb{F} = \{\mathscr{F}_t\}_{0 \le t \le T}$ be the augmented filtration generated by B. Let also $\mathscr{F} = \mathscr{F}_T$. All stochastic processes considered in this paper are assumed to be progressively measurable with respect to $\mathbb{F}$. Let $\mathbb{E}$ denote the expectation operator associated with P.

Let $X : [0, T] \times \Omega \to \mathbb{R}$ be the cumulative output process, which is observable both by the principal and the agent. I employ the *weak formulation*,[1] and assume that

$$X_t = \sigma B_t,$$

with $\sigma > 0$ is a constant. Note that the augmented filtration generated by X coincides with $\mathbb{F}$.

The agent provides effort, which affects the probability measure on $(\Omega, \mathscr{F})$. Let $N : [0, T] \times \Omega \to \mathcal{N}$ denote the process of the agent's effort, where $\mathcal{N} = [0, \overline{N}]$ with $\overline{N} > 0$. The effort process is the agent's private information, and is not observable by the principal.

[1] For the weak formulation, see, for instance, Section 10.4 of Cvitanić and Zhang (2013) and Chapter 9 of Zhang (2017).

The effort process N changes the probability measure on $(\Omega, \mathscr{F})$ from P to P^N in the following way. Define

$$M_t^N \equiv \exp\left(\int_0^t N_s\,dB_s - \frac{1}{2}\int_0^t N_s^2\,ds\right). \tag{1}$$

Since $\mathcal{N}$ is compact, the Novikov condition is satisfied:[2]

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T N_t^2\,dt\right)\right] < \infty, \tag{2}$$

Thus, M^N is a martingale (under P), so that $\mathbb{E}[M_T^N] = 1$. The probability measure induced by the effort process N, P^N, is defined as:

$$dP^N \equiv M_T^N\,dP. \tag{3}$$

By the Girsanov theorem,

$$B_t^N \equiv B_t - \int_0^t N_s\,ds \tag{4}$$

is a standard Brownian motion under P^N. Note that

$$dX_t = \sigma\,dB_t = \sigma N_t\,dt + \sigma\,dB_t^N. \tag{5}$$

Let $\mathbb{E}^N$ be the expectation operator associated with P^N.

At time 0, the principal offers the agent a contract that specifies a wage process $Y : [0,T] \times \Omega \to \mathbb{R}_+$. Since Y is progressively measurable with respect to $\mathbb{F}$, the wage payment at any time t, Y_t, is based (only) on the principal's observation of output until time t, $(X_s; 0 \le s \le t)$. The principal can commit to the contract she offers.

The agent can save/borrow at the (constant) risk-free rate $r > 0$. In addition to the level of effort, the amounts of savings and consumption are the agent's private information, and not observed by the principal. Let $C : [0,T] \times \Omega \to \mathbb{R}_+$ denote the consumption process, and $\tilde{A} : [0,T] \times \Omega \to \mathbb{R}$ the process of the holdings of the risk free bond. Given the contracted wage process Y, they satisfy the flow budget constraint:

$$d\tilde{A}_t = (r\tilde{A}_t + Y_t - C_t)\,dt, \qquad \text{with } \tilde{A}_0 = 0,$$

and the "no-Ponzi-game condition"

$$e^{-rT}\tilde{A}_T \ge 0.$$

[2] For the Girsanov theorem and Novikov condition, see, for instance, Section 3.5 of Karatzas and Shreve (1998).

For the asset position of the agent, I prefer using the discounted value $A_t \equiv e^{-rt}\tilde{A}_t$. Then, the flow budget constraint is given by

$$dA_t = e^{-rt}(Y_t - C_t)\,dt, \qquad \text{with } A_0 = 0, \tag{6}$$

and the no-Ponzi-game condition is by

$$A_T \geq 0. \tag{7}$$

The agent's objective is to maximize the expected utility:

$$\mathbb{E}^N\left[\int_0^T e^{-\rho t}u(C_t, N_t)\,dt\right] \tag{8}$$

where $\rho \geq r$ is the agent's subjective time discount rate. I assume that $u(c, n)$ is bounded; continuously differentiable with bounded derivatives; monotonic: $u_c > 0$, $u_n < 0$; strictly concave in (c, n); and $\lim_{c\to\infty} u_c(c, n) = 0$ for all $n \in \mathcal{N}$.[3] I restrict a pair of consumption and effort processes (C, N) to be $\mathbb{F}$-progressively measurable, $C_t \geq 0$ and $N_t \in \mathcal{N}$ for all $t \in [0, T]$, and $\mathbb{E}\left[\int_0^T (e^{-rt}C_t)^2\ dt\right] < \infty$. It is called *budget-feasible* if the solution A of (6) satisfies (7).

Let $\mathbb{Y}$ be the set of all $\mathbb{F}$-progressively measurable processes Y such that $Y_t \geq 0$ for all $t \in [0, T]$, and

$$\mathbb{E}\left[\int_0^T (e^{-rt}Y_t)^2\ dt\right] < \infty. \tag{9}$$

Given a contract (wage process) $Y \in \mathbb{Y}$, a pair of consumption and effort processes (C, N) is said to be *incentive compatible* if it maximizes the agent's expected utility (8) subject to the budget constraint given by (6)-(7).

The principal's objective is to maximize her expected profit:

$$\mathbb{E}^N\left[\int_0^T e^{-rt}\,(dX_t - Y_t\,dt)\right] = \mathbb{E}^N\left[\int_0^T e^{-rt}(N_t - Y_t)\,dt\right]. \tag{10}$$

In the following subsections, when I discuss the agent's problem, contract $Y \in \mathbb{Y}$ is taken as given and fixed.

2.2 Necessity of the first order conditions

Corresponding to a pair of consumption and effort processes (C, N), the agent's utility process W_t^A is defined as

$$W_t^A \equiv \mathbb{E}_t^N\left[\int_t^T e^{-\rho s}u(C_s, N_s)\,ds\right],$$

[3] The boundedness of u, u_c and u_n is assumed for simplicity, and can be relaxed. What is needed here is the integrability restrictions such as Assumption 10.2.2 in Cvitanić and Zhang (2013).

By the extended martingale representation theorem (e.g., Lemma 10.4.6 in Cvitanić and Zhang (2013)), there exists $(e^{-\rho t} Z_t^A)_{0\le t\le T} \in L^2(\mathbb{F}, P^N)$ such that[4]

$$W_t^A = \int_t^T e^{-\rho s} u(C_s, N_s)\, ds - \int_t^T e^{-\rho s} Z_s^A \, dB_s^N.$$

Since $dB_t^N = dB_t - N_t\, dt$, $(W_t^A, e^{-\rho t} Z_t^A)$ is also viewed as the solution to the following backward stochastic differential equation (BSDE):[5]

$$W_t^A = \int_t^T e^{-\rho s}\big[u(C_s, N_s) + Z_s^A N_s\big]\, ds - \int_t^T e^{-\rho s} Z_s^A \, dB_s. \tag{11}$$

The agent's problem is to choose the consumption and effort processes (C, N) so as to maximize W_0 subject to (11) and (6)-(7).

For each (C, N), the adjoint process for W^A is given by the Radon-Nikodym density M^N defined by (1), and the adjoint process for A is given by the solution $(\Gamma_t, e^{-\rho t} Z_t^\Gamma)$ to the BSDE:

$$d\Gamma_t = e^{-\rho t} Z_t^\Gamma \, dB_t, \qquad \Gamma_T = M_T^N e^{-\rho T} \partial_c u(C_T, N_T). \tag{12}$$

The Hamiltonian is defined as

$$H(t, y, c, n, z^A, m^N, \gamma) \equiv m^N e^{-\rho t}\big[u(c, n) + z^A n\big] + \gamma e^{-rt}(y - c). \tag{13}$$

The first-order conditions for (C, N) are derived from maximizing the Hamiltonian for each t and $(Y_t, Z_t^A, M_t^N, \Gamma_t)$:

$$\max_{c\in\mathbb{R}_+, n\in\mathcal{N}} H(t, Y_t, c, n, Z_t^A, M_t^N, \Gamma_t)$$

Under the assumptions here, as long as $\Gamma_t > 0$, this maximization problem has a unique solution $(C_t, N_t) \in \mathbb{R}_+ \times \mathcal{N}$, which satisfies the following first-order conditions:

$$\partial_c H(t, Y_t, C_t, N_t, Z_t^A, M_t^N, \Gamma_t) \begin{cases} \le 0, & \text{if } C_t = 0, \\ = 0, & \text{if } C_t > 0, \end{cases} \tag{14}$$

$$\partial_n H(t, Y_t, C_t, N_t, Z_t^A, M_t^N, \Gamma_t) \begin{cases} \le 0, & \text{if } N_t = 0, \\ = 0, & \text{if } N_t \in (0, \overline{N}), \\ \ge 0, & \text{if } N_t = \overline{N}. \end{cases} \tag{15}$$

The necessity is standard. Here, it can be proven in (almost) the same way as in Theorem 10.2.5 of Cvitanić and Zhang (2013).

[4] $L^2(\mathbb{F}, P^\ell)$ is the space of $\mathbb{F}$-progressively measurable processes x such that $\|x\|_2 \equiv \left(\mathbb{E}^\ell\left[\int_0^T |x_t|^2 \, dt\right]\right)^{\frac{1}{2}} < \infty$.

[5] For BSDEs, see, for instance, Chapter 4 of Zhang (2017).

Proposition 1. *Let a wage process $Y \in \mathbb{Y}$ be given. Consider a pair of consumption and effort processes $(\hat{C}, \hat{N})$ satisfying the budget constraint, so that the associated wealth process $\hat{A}$ satisfies*

$$d\hat{A}_t = e^{-rt}(Y_t - \hat{C}_t)\,dt, \qquad \hat{A}_0 = 0, \qquad \hat{A}_T \geq 0.$$

Let $(\hat{W}^A, \hat{Z}^A)$ be the solution to (11):

$$d\hat{W}^A_t = -e^{-\rho t}\big[u(\hat{C}_t, \hat{N}_t) + \hat{Z}^A_t \hat{N}_t\big]\,dt + e^{-\rho t}\hat{Z}^A_t\,dB_t, \qquad \hat{W}^A_T = 0,$$

that is, $\hat{W}^A$ is the agent's utility process associated with $(\hat{C}, \hat{N})$. Let $\hat{M}^N$ and $(\hat{\Gamma}, \hat{Z}^\Gamma)$ be the solutions to the SDE and BSDE given, respectively, by:

$$\begin{aligned} d\hat{M}^N_t &= \hat{M}^N_t \hat{N}_t\,dB_t, & \hat{M}^N_0 &= 1,\\ d\hat{\Gamma}_t &= e^{-\rho t}\hat{Z}^\Gamma_t\,dB_t, & \hat{\Gamma}_T &= \hat{M}^N_T e^{(r-\rho)T} u_c(\hat{C}_T, \hat{N}_T). \end{aligned}$$

If $(\hat{C}, \hat{N})$ is incentive compatible for the agent, then the first-order conditions are satisfied:

$$\hat{M}^N_t e^{-\rho t}\partial_c u(\hat{C}_t, \hat{N}_t) - \hat{\Gamma}_t e^{-rt} \begin{cases} \leq 0, & \text{if } \hat{C}_t = 0,\\ = 0, & \text{if } \hat{C}_t > 0, \end{cases}$$

$$\partial_n u(\hat{C}_t, \hat{N}_t) + \hat{Z}^A_t \begin{cases} \leq 0, & \text{if } \hat{N}_t = 0,\\ = 0, & \text{if } \hat{N}_t \in (0, \overline{N}),\\ \geq 0, & \text{if } \hat{N}_t = \overline{N}. \end{cases}$$

and $\hat{A}_T = 0$.

It is useful to define

$$\Lambda_t \equiv \frac{\Gamma_t}{M^N_t}.$$

Then, using the Itô formula, we see that $(\Lambda_t, e^{-\rho^t} Z^\Lambda_t)$ is the solution to the following BSDE:

$$d\Lambda_t = -N_t e^{-\rho t} Z^\Lambda_t\,dt + e^{-\rho t} Z^\Lambda_t\,dB_t; \qquad \Lambda_T = e^{(r-\rho)T}\partial_c u(C_T, N_t). \tag{16}$$

Then the first-order conditions in Proposition 1 are rewritten as:

$$e^{-\rho t}\partial_c u(\hat{C}_t, \hat{N}_t) - \hat{\Lambda}_t e^{-rt} \begin{cases} \leq 0, & \text{if } \hat{C}_t = 0,\\ = 0, & \text{if } \hat{C}_t > 0, \end{cases} \tag{17}$$

$$\partial_n u(\hat{C}_t, \hat{N}_t) + \hat{Z}^A_t \begin{cases} \leq 0, & \text{if } \hat{N}_t = 0,\\ = 0, & \text{if } \hat{N}_t \in (0, \overline{N}),\\ \geq 0, & \text{if } \hat{N}_t = \overline{N}. \end{cases} \tag{18}$$

2.3 Difficulty to prove the sufficiency of the first-order conditions

The Hamiltonian H defined in (13) is strictly concave in (c, n) under our assumption. Then, it might appear to be straightforward to show the validity of the first-order approach using the stochastic maximum principle. Unfortunately, that is not true. It is because to apply the existing theorem on the sufficiency of the stochastic maximum principle (e.g., Theorem 10.2.9 of Cvitanić and Zhang (2013)), we need the Hamiltonian H to be concave in (c, n, z^A). However, because of the multiplicative term nz^A in H, this concavity assumption is not satisfied.[6]

The multiplicative term nz^A is due to the fact that the agent's choice of the effort process N affects the probability distribution of the cumulative output process X. Thus, this difficulty has the same root as the existing literature on the static and discrete-time models encounters on the validity of the first-order approach to the principal-agent model. In the next subsection I prove the validity of the first-order approach based on the stochastic Hamilton-Jacobi-Bellman equation, rather than the stochastic maximum principle.

2.4 Sufficient conditions

To establish the validity of the first-order approach for the principal's problem in the next subsection, I begin by deriving the sufficient conditions for the agent's utility maximization problem based on the dynamic programming approach for *non-Markovian* systems. In particular, I follow the approach described in Section 5.4 of Øksendal and Sulem (2019).

Here, I define the agent's *strategy* by a pair of functions $c(t, a, \omega)$ and $n(t, a, \omega)$, where $c : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}_+$ and $n : [0, T] \times \mathbb{R} \times \Omega \to \mathcal{N}$. I assume that c and n are $\mathbb{F}$-progressively measurable for any fixed a; and c and n are uniformly Lipschitz continuous in a. These assumptions are made so that Assumption 2.1 in Ma et al. (2012) is satisfied.

Notice that the agent's strategy is *not* required to be Markovian because c and n are allowed to depend on $\omega \in \Omega$. In other words, the choice of consumption and effort at each point of time t can depend on the whole history of output $(X_s : 0 \leq s \leq t)$ in addition to time t and the amount of asset A_t.

Associated with a given strategy (c, n), (A, W^A, Z^A) is given by the solution to the forward

[6] The difficulty in applying the sufficiency theorem of the stochastic maximum principle for the principal-agent model has been discussed, for instance, in Williams (2008), and Cvitanić and Zhang (2013).

backward stochastic differential equations (FBSDE):

$$dA_t = e^{-rt}\Big\{Y(t,\omega) - c(t,A_t,\omega)\Big\}\,dt, \qquad A_0 = 0; \tag{19}$$

$$dW_t^A = -e^{-\rho t}\Big\{u\big(c(t,A_t,\omega), n(t,A_t,\omega)\big) + Z_t^A n(t,A_t,\omega)\Big\}\,dt + e^{-\rho t} Z_t^A\,dB_t, \qquad W_T^A = 0. \tag{20}$$

Let $\mathcal{Q}$ be the set of all $(s,a) \in [0,T] \times \mathbb{R}$ such that there exists a strategy (c,n) such that $A_T \geq 0$ a.s., where A is the solution to SDE (19) with initial condition $A_s = a$. Clearly, $\mathcal{Q}$ takes the form that $\mathcal{Q} = \{(t,a) \in [0,T] \times \mathbb{R} : a \geq \underline{a}_t\}$ for some function $\underline{a} : [0,T] \to (-\infty, 0]$.

Given the no-Ponzi-game constraint (7), it might be natural to restrict the domain of strategies (and the associated values) to be $\mathcal{Q} \times \Omega$. However, it turns out to be more convenient to have their domain be $[0,T] \times \mathbb{R} \times \Omega$. For this purpose, I call a strategy (c,n) to be *budget-feasible* if $c(t,a,\omega) = 0$ for all $(t,a) \notin \mathcal{Q}$. Let Π be the set of budget feasible strategies, which depends on the contract Y. This definition of the budget-feasibility is consistent with the no-Ponzi-game constraint in the sense that starting with $(t,a) \in \mathcal{Q}$, $A_T \geq 0$ a.s. for any budget feasible strategy (c,n).

Notice that FBSDE (19)-(20) is *decoupled* in the sense that the SDE for A does not depend on the backward components (W^A, Z^A). Further, it follows from Theorem 8.3.5 in Zhang (2017) that FBSDE (19)-(20) has a *regular decoupling field* $w^A : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ and admits a unique solution.[7] The solution to FBSDE (19)-(20) satisfies

$$W_t^A = w^A(t, A_t, \omega), \qquad \forall t \in [0,T], \text{ a.s.}$$

As shown in Ma et al. (2012), the decoupling field w^A can be characterized as (a part of) the solution to a backward stochastic partial differential equation (BSPDE). To see it heuristically, suppose that for each $a \in \mathbb{R}$, $(w^A, e^{-\rho t} z^A)$ is the solution to the BSDE:

$$dw^A(t,a,\omega) = -G^{c,n}(t,a,\omega)\,dt + e^{-\rho t} z^A(t,a,\omega)\,dB_t, \tag{21}$$

$$w^A(T,a,\omega) = 0, \tag{22}$$

where the function $G^{c,n} : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is to be determined.

By applying the Itô-Ventzell formula,[8] we obtain

$$dw^A(t,A_t,\omega) = \Big\{-G^{c,n}(t,A_t,\omega) + \partial_a w^A(t,A_t,\omega) e^{-rt}\big[Y(t,\omega) - c(t,A_t,\omega)\big]\Big\}\,dt + e^{-\rho t} z^A(t,A_t,\omega)\,dB_t, \tag{23}$$

[7] For the decoupling field, see, for instance, Definition 8.3.2 in Zhang (2017).

[8] See, for instance, Zhang (2017), p. 238.

where $\partial_a w^A(t,a,\omega) \equiv \frac{\partial w^A}{\partial a}(t,a,\omega)$. Comparing (20) and (23), we obtain

$$Z_t^A = z^A(t, A_t, \omega) \tag{24}$$

and

$$\begin{aligned} G^{c,n}(t,a,\omega) = e^{-\rho t}\Big\{u\big[c(t,a,\omega), n(t,a,\omega)\big] + z^A(t,a,\omega)n(t,a,\omega)\Big\} \\ + \partial_a w^A(t,a,\omega)e^{-rt}\big[Y(t,\omega) - c(t,a,\omega)\big] \end{aligned} \tag{25}$$

With $G^{c,n}(t,a,\omega)$ given by (25), the system of equations given by (21)-(22) define a backward stochastic partial differential equation (BSPDE). The unique existence of a (regular weak) solution (w^A, z^A) to this BSPDE is guaranteed by Theorem 6.1 of Ma et al. (2012).

Now we can apply the *stochastic HJB equation* (Theorem 5.16) of Øksendal and Sulem (2019) to show the following proposition.[9]

Proposition 2. *Let a wage process $Y \in \mathbb{Y}$ be given. Consider a budget feasible strategy of the agent $(\hat{c}, \hat{n}) \in \Pi$, and let $(\hat{A}, \hat{W}^A, \hat{Z}^A)$ be the associated solution to FBSDE (19)-(20). Let $(\hat{w}^A, \hat{z}^A)$ be the solution to the BSPDE (21)-(22). Assume that for each (t,a,ω), $(\hat{c}(t,a,\omega), \hat{n}(t,a,\omega))$ is a maximizer of $G^{c,n}(t,a,\omega)$:*

$$\begin{aligned} \big(\hat{c}(t,a,\omega), \hat{n}(t,a,\omega)\big) \in \arg \max_{(c,n)\in\mathbb{R}_+\times\mathcal{N}} \Big\{e^{-\rho t}\big[u(c,n) + \hat{z}^A(t,a,\omega)n\big] \\ + \partial_a \hat{w}^A(t,a,\omega)e^{-rt}\big[Y(t,\omega) - c\big]\Big\} \end{aligned} \tag{26}$$

Then $(\hat{c}, \hat{n})$ is an optimal strategy with optimal value

$$\sup_{(c,n)\in\Pi} W_0^A = \hat{w}^A(0,0,\omega).$$

The first-order conditions associated with the maximization problem in (26) are

$$e^{-\rho t}\partial_c u\big[\hat{c}(t,a,\omega), \hat{n}(t,a,\omega)\big] - \partial_a \hat{w}^A(t,a,\omega)e^{-rt} \begin{cases} \le 0, & \text{if } \hat{c}(t,a,\omega) = 0, \\ = 0, & \text{if } \hat{c}(t,a,\omega) > 0, \end{cases} \tag{27}$$

$$\partial_n u\big[\hat{c}(t,a,\omega), \hat{n}(t,a,\omega)\big] + \hat{z}^A(t,a,\omega) \begin{cases} \le 0, & \text{if } \hat{n}(t,a,\omega) = 0, \\ = 0, & \text{if } \hat{n}(t,a,\omega) \in (0, \overline{N}), \\ \ge 0, & \text{if } \hat{n}(t,a,\omega) = \overline{N}. \end{cases} \tag{28}$$

Let us compare the first-order conditions obtained in the necessity proposition: equations (17)-(18), and those obtained for sufficiency: equations (27)-(28). As in (24), we already know

[9] As discussed in Section 11.3 of Zhang (2017), stochastic HJB equations can be considered as path dependent HJB equations, and one may apply the results on path dependent HJB equations.

that $\hat{Z}_t^A = \hat{z}^A(t, \hat{A}_t, \omega)$ for all $t \in [0, T]$ a.s. Thus, those conditions are equivalent if $\hat{\Lambda}_t = \partial_a \hat{w}^A(t, \hat{A}_t, \omega)$ for all t a.s.

To see it, notice that $(\hat{A}_t, \hat{\Lambda}_t, e^{-\rho t}\hat{Z}_t^\Lambda)$ is the solution to the decoupled FBSDE:

$$\begin{aligned} d\hat{A}_t &= e^{-rt}\Big\{Y(t,\omega) - \hat{c}(t, \hat{A}_t, \omega)\Big\}\, dt, & \hat{A}_0 &= 0; \\ d\hat{\Lambda}_t &= -\hat{n}(t, \hat{A}_t, \omega)e^{-\rho t}\hat{Z}_t^\Lambda + e^{-\rho t}\hat{Z}_t^\Lambda\, dB_t, & \hat{\Lambda}_T &= e^{(r-\rho)T}\partial_c u\big[\hat{c}(T, \hat{A}_T, \omega), \hat{n}(T, \hat{A}_T, \omega)\big]. \end{aligned} \tag{29}$$

This system has a unique regular decoupling field. On the other hand, from the definition of $G^{c,n}$ in (25), it follows that

$$\partial_a G^{\hat{c},\hat{n}}(t, a, \omega) = e^{-\rho t}\partial_a \hat{z}^A(t, a, \omega)\hat{n}(t, a, \omega) + \partial_{aa} w^A(t, a, \omega)\big[Y(t, \omega) - \hat{c}(t, a, \omega)\big] \tag{30}$$

The BSPDE for $\partial_a \hat{w}^A(t, a, \omega)$ is

$$d\big[\partial_a \hat{w}^A(t, a, \omega)\big] = -\partial_a G^{\hat{c},\hat{n}}(t, a, \omega)\, dt + e^{-\rho t}\partial_a \hat{z}^A(t, a, \omega)\, dB_t$$

Then, using the Itô-Ventzell formula,

$$d\big[\partial_a \hat{w}^A(t, \hat{A}_t, \omega)\big] = -e^{-\rho t}\partial_a \hat{z}^A(t, A_t, \omega)\hat{n}(t, \hat{A}_t, \omega)\, dt + e^{-\rho t}\partial_a \hat{z}^A(t, \hat{A}_t, \omega)\, dB_t \tag{31}$$

Comparing equations (29) and (31), we conclude that $\partial_a \hat{w}^A(t, a, \omega)$ is indeed the decoupling field for $\hat{\Lambda}_.$:

$$\hat{\Lambda}_t = \partial_a \hat{w}(t, \hat{A}_t, \omega), \quad \text{and} \quad \hat{Z}_t^\Lambda = \partial_a \hat{z}^A(t, \hat{A}_t, \omega), \qquad \forall t \in [0, T], \quad \text{a.s.}. \tag{32}$$

2.5 Principal's problem: First-order approach

In the previous subsection, it is shown that incentive compatible strategies of the agent are characterized by the stochastic HJB equation (26). Based on that result, the validity of the first-order approach to the principal's problem is established in this subsection.

For this purpose, I assume that for each contract $Y \in \mathbb{Y}$, there exists an incentive compatible strategy of the agent $(c, n) \in \Pi$ satisfying (26). In addition, as is standard in the literature,[10] I restrict attention to those contracts for which the agent chooses to neither save nor borrow:

$$A_t = 0, \qquad \forall t \in [0, T], \text{ a.s.} \tag{33}$$

This condition is equivalent to

$$c(t, 0, \omega) = Y(t, \omega), \qquad \forall t \in [0, T], \text{ a.s.} \tag{34}$$

[10] See, for instance, Williams (2015) and Di Tella and Sannikov (2021).

The zero saving strategy is incentive compatible for the agent if Y and n satisfy

$$e^{-\rho t}\partial_c u\big[Y(t,\omega), n(t,0,\omega)\big] - \partial_a w^A(t,0,\omega)e^{-rt} \begin{cases} \leq 0, & \text{if } Y(t,\omega) = 0, \\ = 0, & \text{if } Y(t,\omega) > 0, \end{cases} \tag{35}$$

$$\partial_n u\big[Y(t,\omega), n(t,0,\omega)\big] + z^A(t,0,\omega) \begin{cases} \leq 0, & \text{if } n(t,0,\omega) = 0, \\ = 0, & \text{if } n(t,0,\omega) \in (0,\overline{N}), \\ \geq 0, & \text{if } n(t,0,\omega) = \overline{N}. \end{cases} \tag{36}$$

Since $A_t = 0$ for all $t \in [0,T]$ a.s. under the zero-saving strategy, the values of $c(t,a,\omega)$ and $n(t,a,\omega)$ for $a \neq 0$ are irrelevant for the principal. Hence, the principal's problem is to maximize her expected profit (10) subject to $N_t = n(t,0,\omega)$; Y and $n(t,0,\omega)$ satisfy conditions (35)-(36), where (w^A, z^A) is the solution to the BSPDE (21)-(22).

Now, recall that w^A is the decoupling field of W^A and $\partial_a w^A$ is that of Λ. It implies the validity of the first-order approach as stated in the next proposition.

Proposition 3. *Given W_0^A, the principal's problem is to choose a contract $Y \in \mathbb{Y}$ so as to maximize her expected profit:*

$$\mathbb{E}^N\left[\int_0^T e^{-rt}(N_t - Y_t)\,dt\right] \tag{37}$$

subject to

$$dW_t^A = -e^{-\rho t}\Big\{u(Y_t, N_t) + Z_t^A N_t\Big\}\,dt + e^{-\rho t}Z_t^A\,dB_t, \qquad W_T^A = 0, \tag{38}$$

$$d\Lambda_t = -N_t e^{-\rho t} Z_t^\Lambda\,dt + e^{-\rho t}Z_t^\Lambda\,dB_t, \qquad \Lambda_T = e^{(r-\rho)T}\partial_c u(Y_T, N_t), \tag{39}$$

and

$$e^{-\rho t}\partial_c u(Y_t, N_t) - \Lambda_t e^{-rt} \begin{cases} \leq 0, & \textit{if } Y_t = 0, \\ = 0, & \textit{if } Y_t > 0, \end{cases} \tag{40}$$

$$\partial_n u(Y_t, N_t) + Z_t^A \begin{cases} \leq 0, & \textit{if } N_t = 0, \\ = 0, & \textit{if } N_t \in (0,\overline{N}), \\ \geq 0, & \textit{if } N_t = \overline{N}. \end{cases} \tag{41}$$

There are different ways to solve the principal's problem, as is described, for instance, in Cvitanić and Zhang (2013). Here, I illustrate the approach based on dynamic programming. For this purpose, let us first rewrite the equations obeyed by W^A and Λ in Proposition 3 as SDEs, rather than BSDEs:

$$W_t^A = W_0^A - \int_0^t e^{-\rho s}\Big\{u\big(Y(s,Z_s^A,\Lambda_s), N(s,Z_s^A,\Lambda_s)\big) + Z_s^A N(s,Z_s^A,\Lambda_s)\Big\}\,ds - \int_0^t e^{-\rho s}Z_s^A\,dB_s,$$

$$\Lambda_t = \Lambda_0 - \int_0^t N(s,Z_s^A,\Lambda_s)e^{-\rho s}Z_s^\Lambda\,dt + \int_0^t e^{-\rho s}Z_s^\Lambda\,dB_s.$$

where $Y(t, z^A, \lambda)$ and $N(t, z^A, \lambda)$ are the solution to (40)-(41). Next, since $dB_t = dB_t^N + N_t\, dt$, these equations are rewritten as

$$W_t^A = W_0^A - \int_0^t e^{-\rho s} u\big(Y(s, Z_s^A, \Lambda_s), N(s, Z_s^A, \Lambda_s)\big)\, ds - \int_0^t e^{-\rho s} Z_s^A\, dB_s^N, \tag{42}$$

$$\Lambda_t = \Lambda_0 + \int_0^t e^{-\rho s} Z_s^\Lambda\, dB_s^N. \tag{43}$$

The principal takes the agent's initial utility W_0^A as given, and commits herself to delivering it to the agent. On the other hand, the initial value of Λ, Λ_0, is chosen by the principal to maximize her expected profit.

The principal's problem is to maximize (37) subject to (42)-(43). Let $V^P(t, w, \lambda)$ be the value function for the principal's problem. The HJB equation is

$$\sup_{z^A, z^\Lambda} \Big\{ e^{-rt}\big[N(t, z^A, \lambda) - Y(t, z^A, \lambda)\big] + \partial_t V^P - \partial_w V^P e^{-\rho t} u\big(Y(t, z^A, \lambda), N(t, z^A, \lambda)\big) + \frac{1}{2}\partial_{ww} V^P (e^{-\rho t} z^A)^2 - \partial_{w\lambda} V^P e^{-2\rho t} z^A z^\Lambda + \frac{1}{2}\partial_{\lambda\lambda} V^P (e^{-\rho t} z^\Lambda)^2 \Big\} = 0 \tag{44}$$

with the terminal condition $V^P(T, w, \lambda) = 0$. For a given initial value W_0^A, the initial value of Λ is determined by maximizing $V^P(0, W_0^A, \cdot)$.

Remark. Proposition 3 might be extended to the case where shocks follow a Markov chain, rather than a Brownian motion. Then it would apply to the optimal unemployment insurance problem considered, for instance, by Hopenhayn and Nicolini (1997), Kocherlakota (2004), Mitchell and Zhang (2010).

3 Hidden returns

In this section, I consider a version of the principal-agent model studied by Di Tella and Sannikov (2021). The key innovation is that I assume neither CRRA preferences for the agent nor the restriction on the volatility of the compensation process assumed by Di Tella and Sannikov (2021).

3.1 The model

The agent manages risky capital delegated by the principal. The instantaneous return of capital reported to (observed by) the principal is

$$dR_t = (r + \alpha - N_t)\, dt + \sigma\, dB_t^N$$

where r is the risk-free rate; $\alpha > 0$ is the risk premium; $\sigma > 0$ is the volatility of the return; $N_t \geq 0$ is the hidden action that the agent takes to divert returns for his private benefits; and B^N is a standard Brownian motion defined by equation (49) below.

As in the previous section, I state the model in the framework of weak formulation. Suppose that $(\Omega, \mathscr{F}, P)$ is a complete probability space, on which a standard Brownian motion B is defined; $\mathbb{F} = \{\mathscr{F}_t\}_{0\leq t\leq T}$ is the augmented filtration generated by B; and $\mathbb{E}$ is the expectation operator associated with P.

The cumulative return process is defined as the strong solution to

$$dR_t = (r+\alpha)\,dt + \sigma\,dB_t$$

The diversion action N affects the probability distribution of the return process. Let $\mathbb{N}$ be the set of $\mathbb{F}$-progressively measurable processes $N : [0,T]\times\Omega \to \mathbb{R}_+$ such that N/σ satisfies the Novikov condition:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \frac{N_t^2}{\sigma^2}\,dt\right)\right] < \infty, \tag{45}$$

and

$$\mathbb{E}\left[\int_0^T e^{-2rt} N_t^4\,dt\right] < \infty. \tag{46}$$

Then, for $N \in \mathbb{N}$, the process M^N defined by

$$M_t^N \equiv \exp\left(-\int_0^t \frac{N_s}{\sigma}\,dB_s - \frac{1}{2}\int_0^t \frac{N_s^2}{\sigma^2}\,ds\right) \tag{47}$$

is a martingale; the probability measure P^N defined by

$$dP^N \equiv M_T^N\,dP \tag{48}$$

is the measure induced by action N; and B^ℓ defined by

$$B_t^N \equiv B_t + \int_0^t \frac{N_s}{\sigma}\,ds \tag{49}$$

is a standard Brownian motion on $(\Omega, \mathscr{F}, P^N)$. The return process R is then expressed as

$$dR_t = (r+\alpha)\,dt + \sigma\,dB_t = (r+\alpha-N_t)\,dt + \sigma\,dB_t^N \tag{50}$$

A contract offered by the principal consists of a pair of processes $(Y,K) \in \mathbb{Y}\times\mathbb{K}$, where Y_t is the wage at time t; K_t is the amount of capital that the agent is delegated to manage at time t; and $\mathbb{K}$ is the set of progressively measurable processes $K : [0,T]\times\Omega \to \mathbb{R}_+$ such that

$$\mathbb{E}\left[\int_0^T e^{-2rt} K_t^4\,dt\right] < \infty.$$

Diversion of $N_t \geq 0$ gives the agent a flow of funds $\phi N_t K_t$, where $\phi \in (0, 1)$.

Without known by the principal, the agent can freely borrow and lend at the risk-free interest rate r, subject to the no-Ponzi-game condition (7). Let $\tilde{A}_t$ be the risk-free asset owned by the agent at time t, and let $A_t \equiv e^{-rt}\tilde{A}_t$. Then, the budget constraint of the agent is given by

$$dA_t = e^{-rt}(Y_t + \phi K_t N_t - C_t)\, dt, \qquad A_0 = 0, \qquad A_T \geq 0, \tag{51}$$

Here $C \in \mathbb{C}$ is the consumption process of the agent, where $\mathbb{C} \equiv \mathbb{Y}$.

Given the contract $(Y, K) \in \mathbb{Y} \times \mathbb{K}$, the agent chooses $(C, N) \in \mathbb{C} \times \mathbb{N}$ so as to maximize his expected utility

$$\mathbb{E}^{\ell}\left[\int_0^T e^{-\rho t} u(C_t)\, dt\right] \tag{52}$$

subject to the budget constraint (51), where $\rho \geq r > 0$. Here, the flow utility function u is a function of C_t. I make similar assumptions as in the previous section: u is bounded; continuously differentiable with bounded derivatives; monotonically increasing; strictly concave; and $\lim_{c\to\infty} u_c(c) = 0$.

A pair (C, N) is said to be *incentive compatible* with respect to (Y, K) if it solves the agent's utility maximization problem.

3.2 Necessity of the first order conditions

Associated with a consumption process $C \in \mathbb{C}$, the agent's utility process W^A is defined as

$$W_t^A \equiv \mathbb{E}_t^N\left[\int_t^T e^{-\rho s} u(C_s)\, ds\right]$$

By the extended martingale representation theorem (Cvitanić and Zhang (2013)), there exists $(e^{-\rho t}\sigma Z_t^A)_{0\leq t\leq T} \in L^2(\mathbb{F}, P^N)$ such that

$$W_t^A = \int_t^T e^{-\rho s} u(C_s)\, ds - \int_t^T e^{-\rho s}\sigma Z_s^A\, dB_s^N$$

Since $dB_t^N = dB_t + N_t/\sigma\, dt$, $(W_t^A, e^{-\rho t}\sigma Z_t^A)$ is the solution to the BSDE:

$$W_t^A = \int_t^T e^{-\rho s}\left[u(C_s) - Z_s^A N_s\right] ds - \int_t^T e^{-\rho s}\sigma Z_s^A\, dB_s. \tag{53}$$

The agent's problem is to choose $(C, N) \in \mathbb{C} \times \mathbb{N}$ to maximize W_0^A subject to (51) and (53).

For a given pair (C, N), the adjoint process for W^A is given by M^N defined by (47), and the adjoint process for A is given by the solution $(\Gamma_t, e^{-\rho t}Z_t^{\Gamma})$ of the BSDE:

$$d\Gamma_t = e^{-\rho t} Z_t^{\Gamma}\, dB_t, \qquad \Gamma_T = M_T^N e^{-\rho T}\partial_c u(C_T).$$

The Hamiltonian is defined by

$$H(t,y,k,c,n,z^A,m^N,\gamma) \equiv m^N e^{-\rho t}\big[u(c) - z^A n\big] + \gamma e^{-rt}(y + \phi k n - c)$$

The first-order conditions for (C, N) are derived from maximizing the Hamiltonian for each t and $(Y_t, K_t, Z_t^A, M_t^N, \Gamma_t)$.

Of particular interest is the condition under which "no diversion," $N = 0$, is an incentive compatible choice of the agent. The next proposition describes a necessary condition for its optimality. Note that when $N = 0$, the adjoint process $M_t^N = 1$ for all t, and thus is dropped from the optimality conditions.

Proposition 4. *Let $(Y, K) \in \mathbb{Y} \times \mathbb{K}$ be a given pair of wage and capital processes. Consider a pair of consumption and no-diversion processes $(\hat{C}, 0) \in \mathbb{C} \times \mathbb{N}$ satisfying the budget constraint, so that the associated wealth process $\hat{A}$ satisfies (51):*

$$d\hat{A}_t = e^{-rt}(Y_t - \hat{C}_t)\,dt, \qquad \hat{A}_0 = 0, \qquad \hat{A}_T \geq 0.$$

The associated utility process $(\hat{W}^A, \hat{Z}^A)$ is the solution to the BSDE (53):

$$d\hat{W}_t^A = -e^{-\rho t} u(\hat{C}_s)\,ds + e^{-\rho t}\sigma Z_t^A\,dB_t, \qquad \hat{W}_T^A = 0.$$

The adjoint process for $\hat{\alpha}$ is given by the solution $(\hat{\lambda}, \hat{Z}^\lambda)$ to the BSDE:

$$d\hat{\Gamma}_t = e^{-\rho t}\hat{Z}_t^\Gamma\,dB_t, \qquad \hat{\Gamma}_T = e^{-\rho T}u_c(\hat{C}_T).$$

Consider the Hamiltonian:

$$H(t,y,k,c,n,z^A,1,\gamma) \equiv e^{-\rho t}\big[u(c) - z^A N\big] + \gamma e^{-rt}(y + \phi k n - c)$$

Then, necessary conditions for $(\hat{C}, 0)$ to be incentive compatible for the agent are given by the first-order conditions

$$e^{-\rho t}u_c(\hat{C}_t) - \hat{\Gamma}_t e^{-rt} \begin{cases} \leq 0, & \text{if } \hat{C}_t = 0, \\ = 0, & \text{if } \hat{C}_t > 0, \end{cases}$$
$$-e^{-\rho t}\hat{Z}_t^A + \hat{\Gamma}_t e^{-rt}\phi K_t \leq 0.$$

and the transversality condition: $\hat{A}_T = 0$.

3.3 Sufficient condition

Just as in the hidden-effort model discussed in the previous section, the sufficiency theorem of the stochastic maximum principle (e.g. Theorem 10.2.9 of Cvitanić and Zhang (2013)) is not applicable because of the term $z^A N$ in the Hamiltonian. Instead, again, I employ the dynamic programming approach.

A contract $(Y, K) \times \mathbb{Y} \times \mathbb{K}$ is given and fixed. The agent's strategy is defined by a pair of functions $c(t, a, \omega)$ and $n(t, a, \omega)$, where $c : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}_+$ and $n : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}_+$, such that c and n are $\mathbb{F}$-progressively measurable for any fixed a, and that c and n are uniformly Lipschitz continuous in a.

Associated with a strategy (c, n), (A, W^A, Z^A) is given by the (decoupled) FBSDE:

$$dA_t = e^{-rt}\Big\{Y(t, \omega) + \phi K(t, \omega) n(t, A_t, \omega) - c(t, A_t, \omega)\Big\}\, dt, \qquad A_0 = 0, \tag{54}$$

$$dW_t^A = -e^{-\rho t}\Big\{u(C_t) - Z_t^A N_t\Big\}\, dt + e^{-\rho t}\sigma Z_t^A\, dB_t, \qquad W_T^A = 0. \tag{55}$$

It has a regular decoupling field $w^A : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ and admits a unique solution such that

$$W_t^A = w^A(t, A_t, \omega), \qquad \forall t \in [0, T], \text{ a.s.}$$

Let Q be the set of all $(s, a) \in [0, T] \times \mathbb{R}$ such that there exists a strategy (c, n) such that $A_T \geq 0$ a.s. A strategy is called budget-feasible if $c(t, a, \omega) = 0$ for all $(t, a) \notin Q$. Let Π be the set of budget-feasible strategies.

As in the previous section, the BSPDE for the decoupling field w^A is derived as

$$dw^A(t, a, \omega) = -G^{c,n}(t, a, \omega)\, dt + e^{-\rho t} z^A(t, a, \omega)\, dB_t \tag{56}$$

$$w^A(T, a, \omega) = 0, \tag{57}$$

where

$$\begin{aligned} G^{c,n}(t, a, \omega) \equiv e^{-\rho t}\Big\{u\big[c(t, a, \omega)\big] - z^A(t, a, \omega) n(t, a, \omega)\Big\} \\ + \partial_a w^A(t, a, \omega) e^{-rt}\Big\{Y(t, \omega) + \phi K(t, \omega) n(t, a, \omega) - c(t, a, \omega)\Big\} \end{aligned} \tag{58}$$

Then we can apply the stochastic HJB equation (Theorem 5.16) of Øksendal and Sulem (2019).

Proposition 5. *Let a contract $(Y, K) \in \mathbb{Y} \times \mathbb{K}$ be given. Consider a budget feasible strategy of the agent $(\hat{c}, 0) \in \Pi$, and let $(\hat{A}, \hat{W}^A, \hat{Z}^A)$ be the associated solution to FBSDE (54)-(55). Let $(\hat{w}^A, \hat{z}^A)$ be the solution to the BSPDE (56)-(57). Assume that for each (t, a, ω), $(\hat{c}(t, a, \omega), 0)$ is a maximizer of $G^{c,n}(t, a, \omega)$:*

$$\begin{aligned} \big(\hat{c}(t, a, \omega), 0\big) \in \arg \max_{(c,n) \in \mathbb{R}_+ \times \mathcal{N}} \Big\{ e^{-\rho t}\big[u(c, n) - \hat{z}^A(t, a, \omega) n\big] \\ + \partial_a \hat{w}^A(t, a, \omega) e^{-rt}\big[Y(t, \omega) + \phi K(t, \omega) n - c\big]\Big\} \end{aligned} \tag{59}$$

Then $(\hat{c}, 0)$ *is an optimal strategy with optimal value*

$$\sup_{(c,n)\in\Pi} W_0^A = \hat{w}^A(0,0,\omega).$$

3.4 Principal's problem

Given a promised level of initial utility of the agent, W_0^A. the principal's objective is to minimize the expected cost of delivering W_0^A to the agent:

$$\mathbb{E}^N\left[\int_0^T e^{-rt}(C_t - \alpha K_t)\right]$$

where α is the risk premium in (50).

Again, I restrict attention to contracts such that

$$c(t,0,\omega) = Y(t,\omega), \qquad \forall t \in [0,T], \text{ a.s.}$$

so that

$$A_t = 0, \qquad \forall t \in [0,T], \text{ a.s.}$$

Then the following proposition follows.

Proposition 6. *Given* W_0^A*, the principal's problem is to choose a contract* $(Y,K) \in \mathbb{Y} \times \mathbb{K}$ *so as to minimize the expected cost:*

$$\mathbb{E}^N\left[\int_0^T e^{-rt}(Y_t - \alpha K_t)\, dt\right] \tag{60}$$

subject to

$$dW_t^A = -e^{-\rho t}\Big\{u(Y_t) - Z_t^A N_t\Big\}\, dt + e^{-\rho t} Z_t^A\, dB_t, \qquad W_T^A = 0, \tag{61}$$

$$d\Gamma_t = e^{-\rho t} Z_t^\Gamma\, dB_t, \qquad \Gamma_T = e^{(r-\rho)T}\partial_c u(Y_T), \tag{62}$$

and

$$e^{-\rho t}\partial_c u(Y_t) - \Gamma_t e^{-rt} \begin{cases} \leq 0, & \text{if } Y_t = 0, \\ = 0, & \text{if } Y_t > 0, \end{cases} \tag{63}$$

$$-e^{-\rho t} Z_t^A + \Gamma_t e^{-rt}\phi K_t \leq 0. \tag{64}$$

4 Conclusion

In this paper, I establish the validity of the first-order approach for the principal-agent problems with hidden saving in continuous time. The principal offers a contract to the agent, where the payment to the agent at each point in time may depend on the history of observable state in an arbitrary way. It make the agent's optimization problem non-Markovian. I use the theory of stochastic HJB equations to characterize the optimality condition for the agent. Then, without loss of generality, I focus on wage contracts for which the agent chooses zero savings. Given this, I show that the principal's optimization problem can be expressed as maximizing her expected profit subject to two SDEs: one equation describing the agent's continuation utility process, and the other being the Euler equation concerning the agent's marginal utility process. My result is an extension of those obtained by Williams (2015) and Di Tella and Sannikov (2021).

References

Biais, B., T. Mariotti, G. Plantin, and J.-C. Rochet (2007): "Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications," *Review of Economic Studies*, 74, 345–390.

Cvitanić, J. and J. Zhang (2013): *Contract Theory in Continuous-Time Models*, Springer Finance, Berlin, Heidelberg: Springer Berlin Heidelberg.

DeMarzo, P. M. and Y. Sannikov (2006): "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model," *The Journal of Finance*, 61, 2681–2724.

Di Tella, S. and Y. Sannikov (2021): "Optimal Asset Management Contracts With Hidden Savings," *Econometrica*, 89, 1099–1139.

Holmstrom, B. and P. Milgrom (1987): "Aggregation and Linearity in the Provision of Intertemporal Incentives," *Econometrica*, 55, 303.

Hopenhayn, H. A. and J. P. Nicolini (1997): "Optimal Unemployment Insurance," *Journal of Political Economy*, 105, 412–438.

Karatzas, I. and S. E. Shreve (1998): *Brownian Motion and Stochastic Calculus*, vol. 113 of *Graduate Texts in Mathematics*, New York, NY: Springer New York.

Kocherlakota, N. R. (2004): "Figuring out the Impact of Hidden Savings on Optimal Unemployment Insurance," *Review of Economic Dynamics*, 7, 541–554.

Ma, J., H. Yin, and J. Zhang (2012): "On Non-Markovian Forward–Backward SDEs and Backward Stochastic PDEs," *Stochastic Processes and their Applications*, 122, 3980–4004.

Mitchell, M. and Y. Zhang (2010): "Unemployment Insurance with Hidden Savings," *Journal of Economic Theory*, 145, 2078–2107.

Øksendal, B. and A. Sulem (2019): *Applied Stochastic Control of Jump Diffusions*, Universitext, Cham: Springer International Publishing.

Peng, S. (1992): "Stochastic Hamilton–Jacobi–Bellman Equations," *SIAM Journal on Control and Optimization*, 30, 284–304.

Sannikov, Y. (2008): "A Continuous-Time Version of the Principal–Agent Problem," *Review of Economic Studies*, 75, 957–984.

Williams, N. (2008): "On Dynamic Principal-Agent Problems in Continuous Time," Working Paper, University of Wisconsin.

——— (2015): "A Solvable Continuous Time Dynamic Principal–Agent Model," *Journal of Economic Theory*, 159, 989–1015.

Zhang, J. (2017): *Backward Stochastic Differential Equations*, vol. 86 of *Probability Theory and Stochastic Modelling*, New York, NY: Springer New York.