Interim Rationalizable (and Bayes-Nash) Implementation of Functions: A full Characterization*

R Jain[†] M Lombardi[‡]

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Abstract

Interim Rationalizable Monotonicity, due to Oury and Tercieux (2012), fully characterizes the class of social choice functions that are implementable in interim correlated rationalizable (and Bayes-Nash equilibrium) strategies.

JEL classification: C79, D82

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[†]Institute of Economics, Academia Sinica, e-mail: ritesh@econ.sinica.edu.tw.

[‡]University of Liverpool Management School. E-mail: michele.lombardi@liverpool.ac.uk. Department of Economics and Statistics and CESF, University of Napoli Federico II.

I. Introduction

II. The Implementation Model

Preliminaries

Throughout the paper, if X is a topological space, we treat it as a measurable space with its Borel sigma field, and the space of Borel probability measures on X is denoted by $\Delta(X)$. Spaces $\Delta(X)$ are endowed with the topology of weak convergence of measures. Throughout the paper, we treat each countable set as a topological space endowed with the discrete topology. A subset Y of a topological space X is a dense subset of X if the closure of Y in X is equal to X. With abuse of notation, given a space X, let δ_x denote a degenerate distribution in $\Delta(X)$ assigning probability 1 to $\{x\}$.

We consider a finite set of players $\mathcal{I} = \{1, ..., I\}$. Each player i has a bounded utility function $u_i : \Delta(A) \times \Theta \to \mathbb{R}$ where A is the set of (pure) outcomes and Θ is the set of states (of nature). For each $\theta \in \Theta$, $u_i(\cdot, \theta)$ satisfies the expected utility hypothesis. We assume that Θ and A are separable metric spaces.

Throughout the paper, if, for each $i \in \mathcal{I}$, there is a space X_i, we write X as an abbreviation for $\Pi_{i\in\mathcal{I}}X_i$ and, for each $i \in \mathcal{I}$, X_{-i} for $\Pi_{j\in\mathcal{I}\setminus\{i\}}X_j$.

A *model* (of incomplete information) is a pair $\mathcal{T} \equiv (T, \kappa)$, where $T = \Pi_{i\in\mathcal{I}}T_i$ is a countable type space and, for each $i \in \mathcal{I}$, $\kappa(t_i) \in \Delta(\Theta \times T_{-i})$ denotes the associated beliefs for each type $t_i \in T_i$ of player i satisfying the following condition: For all $t_i \in T_i$, Supp$(\kappa(t_i)) = \Delta(\Theta \times T_{-i})$.

A typical type profile of T (*resp.*, T_{-i}) is denoted by t (*resp.*, t_{-i}). Throughout the paper, we rule out the case that $\mathcal{T}$ is a model of complete information, for the sake of simplicity.

A (stochastic) *mechanism* is a pair $\mathcal{M} \equiv (M, g)$, where $M \equiv_{i\in\mathcal{I}} M_i$ is a message space and the outcome function $g : M \to \Delta(A)$ assigns to every $m \in M$ an element of $\Delta(A)$. For each $i \in \mathcal{I}$, M_i is player i's message space, which is assumed to be a (nonempty) countable set. A message profile $m \in M$ is often written as (m_i, m_{-i}), where $m_{-i} \in M_{-i}$.

Given a mechanism $\mathcal{M}$ and a model $\mathcal{T}$, $U(\mathcal{M},\mathcal{T})$ denotes the induced game of incomplete information. In this game, a (behavioral) strategy of player i is any measurable function $\sigma_i : T_i \to \Delta(M_i)$. We write $\sigma_i(t_i)[m_i]$ for the probability that σ_i assigns to m_i when player i is of type t_i. Player i's strategy σ_i is a pure strategy if $\sigma_i : T_i \to M_i$. Given a mechanism $\mathcal{M}$, for each $i \in \mathcal{I}$, player i's best response correspondence BR_i from $\Delta(\Theta \times M_{-i})$ to M_i be defined, for all $\pi_i \in \Delta(\Theta \times M_{-i})$, by

$$BR_i(\pi_i|\mathcal{M}) = \arg\max_{m_i \in M_i} \left(\sum_{(\theta, m_{-i}) \in \Theta \times M_{-i}} \pi_i[\theta, m_{-i}] \left[u_i(g(m_i, m_{-i}), \theta) \right] \right).$$

Since we allow for infinite mechanisms, the correspondence may be empty. For all $i \in I$, all $t_i \in T_i$ and all $\sigma_{-i} \equiv (\sigma_j)_{j \in \mathcal{I} \backslash \{i\}}$, we write $\pi_i(t_i, \sigma_{-i}) \in \Delta(\Theta \times M_{-i})$ for the joint distribution on the underlying uncertainty and the messages of other players induced by t_i and σ_{-i}.[1]

Definition 1. Let $U(\mathcal{M},\mathcal{T})$ be any game of incomplete information. A profile of pure strategies $\sigma = (\sigma_i)_{i \in \mathcal{I}}$ is a pure strategy *Bayes-Nash equilibrium* of $U(\mathcal{M},\mathcal{T})$ if, for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$m_i \in \text{Supp}(\sigma_i(t_i)) \implies m_i \in BR_i(\pi_i(t_i, \sigma_{-i})|\mathcal{M}).$$

We denote by $BNE(U(\mathcal{M},\mathcal{T}))$ the set of all pure strategy Bayes-Nash equilibria of $U(\mathcal{M},\mathcal{T})$.

Next, let us define the solution concept of interim correlated rationalizability (ICR, henceforth), which was introduced by Dekel et al. (2007). Before introducing it, we need additional notation. Fix any pair $(\mathcal{M},\mathcal{T})$. For all $i \in \mathcal{I}$, let Σ_i be a nonempty correspondence from T_i to $2^{M_i} \backslash \{\emptyset\}$, and let $\mathfrak{S}_i^{\mathcal{M},\mathcal{T}}$ denote the set of all nonempty correspondences from T_i to $2^{M_i} \backslash \{\emptyset\}$. Let $\mathfrak{S}^{\mathcal{M},\mathcal{T}} = \Pi_{i \in \mathcal{I}} \mathfrak{S}_i^{\mathcal{M},\mathcal{T}}$, with Σ as a typical

[1] Formally, $\pi_i(t_i, \sigma_{-i}) \in \Delta(\Theta \times M_{-i})$ is defined by $\pi_i(t_i, \sigma_{-i}) = \sum_{t_{-i} \in T_{-i}} \kappa(t_i)[\theta, t_{-i}] \sigma_{-i}(t_{-i})[m_{-i}]$, where $\kappa(t_i)[\theta, t_{-i}]$ is the probability attached to $[\theta, t_{-i}]$ under $\kappa(t_i)$, and $\sigma_{-i}(t_{-i})[m_{-i}]$ is the probability attached to m_{-i} under $\sigma_{-i}(t_{-i})$.

profile of $\mathfrak{S}^{\mathcal{M},\mathcal{T}}$. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i})$ be defined by

$$\Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) = \left\{\pi_i \in \Delta(\Theta \times T_{-i} \times M_{-i}) \,|\mathrm{marg}_{\Theta \times T_{-i}}\pi_i = \kappa(t_i)\right\},$$

and, for all $\Sigma_{-i} \in \mathfrak{S}_{-i}^{\mathcal{M},\mathcal{T}}$, let $\Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$ be defined by

$$\Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i}) = \left\{ \pi_i \;\middle|\; \begin{array}{l} \pi_i \in \Delta(\Theta \times T_{-i} \times M_{-i}) \text{ and} \\ \pi_i[\theta, t_{-i}, m_{-i}] > 0 \implies m_{-i} \in \Sigma_{-i}(t_{-i}) \end{array} \right\}.$$

For all $(\mathcal{M}, \mathcal{T})$ and all $\Sigma \in \mathfrak{S}^{\mathcal{M},\mathcal{T}}$, Σ is a *best-reply set* in $U(\mathcal{M}, \mathcal{T})$ if, for all $i \in \mathcal{I}$, all $t_i \in T_i$ and all $m_i \in \Sigma_i(t_i)$, there exists

$$\pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$$

such that

$$m_i \in BR_i\left(\mathrm{marg}_{\Theta \times M_{-i}}\pi_i | \mathcal{M}\right).$$

Definition 2. For all $(\mathcal{M}, \mathcal{T})$, all $i \in \mathcal{I}$ and all $t_i \in T_i$, the *set of interim correlated rationalizable messages at type* t_i, denoted by $S_i^{\mathcal{M},\mathcal{T}}(t_i)$, is defined by

$$S_i^{\mathcal{M},\mathcal{T}}(t_i) = \{m_i \in \Sigma_i(t_i) \,|\text{for some best-reply set } \Sigma \text{ in } U(\mathcal{M}, \mathcal{T})\}.$$

For all $t \in T$, we write $S^{\mathcal{M},\mathcal{T}}(t)$ for $\Pi_{i \in \mathcal{I}} S_i^{\mathcal{M},\mathcal{T}}(t_i)$.

Alternatively, we can compure the set of interim correlated rationalizable strategies interatively as follows. For all i and all $t_i \in T_i$, let $S_i^{0,\mathcal{M},\mathcal{T}}(t_i) = M_i$, and for all integers $k \geq 1$, let $S_i^{k,\mathcal{M},\mathcal{T}}(t_i)$ be defined by

$$S_i^{k,\mathcal{M},\mathcal{T}}(t_i) = \left\{ m_i \in S_i^{k-1,\mathcal{M},\mathcal{T}}(t_i) \;\middle|\; \begin{array}{l} \text{There exists } \pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \\ \text{such that } \pi_i \in \Delta^{S_{-i}^{k-1,\mathcal{M},\mathcal{T}}}(\Theta \times T_{-i} \times M_{-i}) \\ \text{and that } m_i \in BR_i\left(\mathrm{marg}_{\Theta \times M_{-i}}\pi_i | \mathcal{M}\right) \end{array} \right\} \tag{1}$$

Let $\mathcal{T}$ be given. A (stochastic) *social choice function* (SCF, henceforth) is a function $f : T \rightarrow \Delta(A)$. Following Oury and Tercieux (2012), we assume that the planner cares about all profiles of types in T. To avoid trivialities, throughout the paper, we focus on minimally responsive SCFs.

Definition 3. Let $\mathcal{T}$ be any model. $f : T \rightarrow \Delta(A)$ is *minimally responsive on* T if for all $i \in \mathcal{I}$, there exist $t_{-i} \in T_{-i}$ and $t_i, t_i' \in T_i$ such that $f(t_i, t_{-i}) \neq f(t_i', t_{-i})$.

If f is minimally responsive on T, f is a nonconstant SCF.[2] Moreover, if f is not a minimally responsive SCF on T and the planner's objective is to implement f, he can, equivalently, focus on the implementation of $f' : \Pi_{i \in \mathcal{I} \setminus \mathcal{I}^*} T_i \rightarrow \Delta(A)$, where $\mathcal{I}^*$ is the of players for whom f is not a minimally responsive SCF on T, and φ is defined, for all $t \in \Pi_{i \in \mathcal{I} \setminus \mathcal{I}^*} T_i$, by $\varphi(t) = f(t, t')$ for all $t' \in \Pi_{i \in \mathcal{I}^*} T_i$.[3]

Definition 4. A mechanism $\mathcal{M}$ *implements* $f : T \rightarrow \Delta(A)$ *in interim correlated rationalizable strategies* (ICR-implements, henceforth) on $\mathcal{T}$ if the following two conditions are satisfied.

(i) For all $i \in \mathcal{I}$ and all $t_i \in T_i$, $S_i^{\mathcal{M},\mathcal{T}}(t_i) \neq \emptyset$.

(ii) For all $t \in T$, $m \in S^{\mathcal{M},\mathcal{T}}(t) \implies g(m) = f(t)$.

If such a mechanism exists, f is *interim correlated rationalizably* (ICR, henceforth) *implementable*, or simply, *ICR-implementable* on $\mathcal{T}$.

A mechanism $\mathcal{M}$ *implements* $f : T \rightarrow \Delta(A)$ on $\mathcal{T}$ *in (pure strategy) Bayes-Nash equilibria* if $BNE(U(\mathcal{M}, \mathcal{T})) \neq \emptyset$ and for all $\sigma \in BNE(U(\mathcal{M}, \mathcal{T}))$, $g \circ \sigma = f$. Moreover, a mechanism $\mathcal{M}$ satisfies the *Equilibrium Best-Response Property* (EBRP)

[2] f is constant if for all $t, t' \in T$, $f(t) = f(t')$.

[3] Formally, $\mathcal{I}^*$ can be defined as follows:

$$\mathcal{I}^* = \{i \in \mathcal{I} | f(t_i, t_{-i}) = f(t_i', t_{-i}) \text{ for all } t_i, t_i' \in T_i \text{ and all } t_{-i} \in T_{-i}\}$$

on $\mathcal{T}$ if there exists a pure strategy profile σ such that for all $t \in T$,

$$\sigma(t) \in S^{\mathcal{M},\mathcal{T}}(t),$$

and for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$BR_i(\pi_i(t_i, \sigma_{-i}) | \mathcal{M}) \neq \emptyset.$$

Any SCF that is ICR-implementable on $\mathcal{T}$ by a mechanism satisfying the EBRP is also implementable on $\mathcal{T}$ in Bayes-Nash equilibria.

Lemma 1. Assume that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. $\mathcal{M}$ implements f on $\mathcal{T}$ in Bayes-Nash equilibria if and only if $\mathcal{M}$ satisfies the EBRP.

Proof. Assume that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. Assume that $\mathcal{M}$ satisfies the EBRP on $\mathcal{T}$. Let us show that $\mathcal{M}$ implements f on $\mathcal{T}$ in Bayes-Nash equilibria. To this end, we need only to show that $BNE(U(\mathcal{M},\mathcal{T})) \neq \emptyset$. Since $\mathcal{M}$ ICR-implements f and $\mathcal{M}$ satisfies the EBRP, it follows that there exists a pure strategy profile σ such that for all $t \in T$, $\sigma(t) \in S^{\mathcal{M},\mathcal{T}}(t)$, and for all $i \in \mathcal{I}$ and all $t_i \in T_i$, $BR_i(\pi_i(t_i, \sigma_{-i}) | \mathcal{M}) \neq \emptyset$. Let us show that $\sigma \in BNE(U(\mathcal{M},\mathcal{T}))$.

For all $i \in \mathcal{I}$ and all $t_i \in T_i$, since $BR_i(\pi_i(t_i, \sigma_{-i}) | \mathcal{M}) \neq \emptyset$, let $\hat{\sigma}_i(t_i) \in BR_i(\pi_i(t_i, \sigma_{-i}) | \mathcal{M})$ for all $t_i \in T_i$ and all $i \in \mathcal{I}$. Fix any $i \in \mathcal{I}$. By construction, we see that for all $t \in T$, $(\hat{\sigma}_i(t_i), \sigma_{-i}(t_{-i})) \in S^{\mathcal{M},\mathcal{T}}(t)$. Moreover, since $\mathcal{M}$ ICR-implements f on $\mathcal{T}$, we also have that for all $t \in T$, $f(t) = g(\hat{\sigma}_i(t_i), \sigma_{-i}(t_{-i}))$. Thus, we can replace $\hat{\sigma}_i$ with σ_i and see that $\sigma_i(t_i) \in BR_i(\pi_i(t_i, \sigma_{-i}) | \mathcal{M})$ for all $t_i \in T_i$. Since the choice of i was arbitrary, we have that $\sigma \in BNE(U(\mathcal{M},\mathcal{T}))$.

For the converse, assume that $\mathcal{M}$ implements f on $\mathcal{T}$ in Bayes-Nash equilibria. This implies that $BNE(U(\mathcal{M},\mathcal{T})) \neq \emptyset$. Thus, $\mathcal{M}$ satisfies the EBRP on $\mathcal{T}$. □

III. Interim Iterative Monotonicity

In the following section, we present our necessary condition. Let $\mathcal{T}$ be any model. For every player $i \in \mathcal{I}$, let us call any map $\beta_i : T_i \to 2^{T_i} \backslash \{\emptyset\}$ as player i's deception. A special deception for player i is the truth-telling deception, β_i^t, defined by $\beta_i^t(t_i) = \{t_i\}$ for all $t_i \in T_i$. Another special deception for player i is denoted by $\bar{\beta}_i$ and defined by $\bar{\beta}_i(t_i) = T_i$. For any β_i and β_i' we write $\beta_i \subseteq \beta_i'$ if $\beta_i(t_i) \subseteq \beta_i'(t_i)$ for all $t_i \in T_i$. Let $\mathcal{B}_i$ be the set of all player i's deceptions containing the truth-telling deception; that is,

$$\mathcal{B}_i = \left\{\beta_i : T_i \to 2^{T_i} \backslash \{\emptyset\} \, | \beta_i^t \subseteq \beta_i \right\}.$$

Let $\mathcal{B}^t = \Pi_{i\in\mathcal{I}}\mathcal{B}_i^t$, with $\beta = (\beta_i)_{i\in\mathcal{I}}$ as a typical deception profile of $\mathcal{B}$.

For every $i \in \mathcal{I}$, let Y_i^f be the set of mappings from T_{-i} to $\Delta(A)$ satisfying the following requirement. Whatever is player i's actual type, he would never prefer the outcome to be selected by a mapping Y_i^f to the outcome he could obtain under f if all his opponents were reporting truthfully. Formally,

$$Y_i^f = \left\{ y : T_{-i} \to \Delta(A) \;\middle|\; \begin{array}{l} \text{For all } \tilde{t}_i \in T_i, \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa\left(\tilde{t}_i\right)[\theta, t_{-i}]\, u_i\left(f\left(\tilde{t}_i, t_{-i}\right), \theta\right) \geq \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa\left(\tilde{t}_i\right)[\theta, t_{-i}]\, u_i\left(y\left(t_{-i}\right), \theta\right). \end{array} \right\} \quad (2)$$

Note that Y_i^f is a metrizable separable space.[4] We write Y^f for $\Pi_{i\in\mathcal{I}}Y_i^f$. For all $i \in \mathcal{I}$, let $Y_{i,s}^f$ be the set of all mappings in Y_i^f satisfying the inequality in (2) strictly for all $\tilde{t}_i \in T_i$.[5] Similarly, we write Y_s^f for $\Pi_{i\in\mathcal{I}}Y_{i,s}^f$.

For the sake of clarity, in what follows, for every $i \in \mathcal{I}$, we write $T_{-i} \times \hat{T}_{-i}$ for $T_{-i} \times T_{-i}$. In the context of a mechanism, our interpretation of $\left(t_{-i}, \hat{t}_{-i}\right) \in T_{-i} \times \hat{T}_{-i}$ is that player i's opponents are of types t_{-i} but they are playing as if they were of types $\hat{t}_{-i}$.

[4] To see it, observe that $\Delta(A)$ is a separable metric space under the Prohokorov metric given that A is a separable metric space Aliprantis and Border (2006); Theorem 14.15). Moreover, a countable product of the space $\Delta(A)$ is separable metric space under the standard metric (see, e.g., Ok (2011), p. 196). Thus, since Y_i^f is a subset of a separable metric space, it follows that it is a separable metric space.

[5] Formally, for all $i \in \mathcal{I}$,

$$Y_{i,s}^f = \left\{ y : T_{-i} \to \Delta(A) \;\middle|\; \begin{array}{l} \text{For all } \tilde{t}_i \in T_i, \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa\left(\tilde{t}_i\right)[\theta, t_{-i}]\, u_i\left(f\left(\tilde{t}_i, t_{-i}\right), \theta\right) > \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa\left(\tilde{t}_i\right)[\theta, t_{-i}]\, u_i\left(y\left(t_{-i}\right), \theta\right). \end{array} \right\}$$

A deception profile $\beta \in \mathcal{B}$ is *acceptable* on $\mathcal{T}$ for f if for all $t, t' \in T$, $t' \in \beta(t) \Longrightarrow f(t) = f(t')$. The following condition is due to Oury and Tercieux (2012).

Definition 5. $f : T \to \Delta(A)$ is *interim (correlated) rationalizable monotonic* (IRM, henceforth) on $\mathcal{T}$ if for every unacceptable deception profile $\beta \in \mathcal{B}$ on $\mathcal{T}$ for f, there exists $(i, t_i, t_i') \in \mathcal{I} \times T_i \times \beta_i(t_i)$ such that for all $\psi_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, there exists $y_i^* \in Y_i^f$ such that

$$\begin{array}{cc} \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\psi_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(y_i^*\left(\hat{t}_{-i}\right),\theta\right) & > \\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\psi_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(f\left(t_i',\hat{t}_{-i}\right),\theta\right). & \end{array} \tag{3}$$

Remark 1. Observe that if f is IRM on $\mathcal{T}$, then it is strict IRM on $\mathcal{T}$. f is strict IRM on $\mathcal{T}$ if $y^* \in Y_i^f$ satisfying (3) is such that it satisfies the inequality in (2) strictly for $t_i' = \tilde{t}_i$. However, it can be shown that the two conditions are equivalent (see Supplementary Appendix).

A condition, which is equivalent to IRM, can be expressed in terms of the limit point of an iterative sequence of deception profiles. To define the sequence, we need additional notation. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be defined by

$$\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) = \left\{\nu_i \in \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) | \mathrm{marg}_{\Theta\times T_{-i}}\nu_i = \kappa(t_i)\right\},$$

and, moreover, for all $\beta \in \mathcal{B}$, let $\Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be defined by

$$\Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) = \left\{ \nu_i \;\middle|\; \begin{array}{l} \nu_i \in \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \text{ and} \\ \nu_i\left[\theta, t_{-i}, \hat{t}_{-i}\right] > 0 \implies \hat{t}_{-i} \in \beta_{-i}(t_{-i}) \end{array} \right\}.$$

The iterative sequence defined on Y^f, denoted by $\left(\beta^k\right)_{k\geq 0}$, is defined as follows. The starting value is

$$\beta^0 = \bar{\beta},$$

and, for all $k \geq 1$ and all $i \in \mathcal{I}$, β_i^k is defined, for all $t_i \in T_i$, by

$$\beta_i^k(t_i) = \left\{ \hat{t}_i \;\middle|\; \begin{array}{l} \hat{t}_i \in \beta_i^{k-1}(t_i) \text{ and there exists} \\ \nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \text{ such} \\ \text{that } \nu_i(t_i) \in \Delta^{\beta_{-i}^{k-1}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \text{ and} \\ \sum_{(\theta,\hat{t}_{-i}) \in \Theta \times \hat{T}_{-i}} \left(\text{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i)\left[\theta, \hat{t}_{-i}\right]\right) u_i\left(f\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) \geq \\ \sum_{(\theta,\hat{t}_{-i}) \in \Theta \times \hat{T}_{-i}} \left(\text{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i)\left[\theta, \hat{t}_{-i}\right]\right) u_i\left(y_i\left(\hat{t}_{-i}\right), \theta\right), \\ \text{for all } y_i \in Y_i^f. \end{array} \right\} \quad (4)$$

Observe that $t_i \in \beta_i^k(t_i)$ for all $i \in \mathcal{I}$, all $t_i \in T_i$ and all $k \geq 0$. If the limit point of the sequence $\left(\beta^k\right)_{k \geq 0}$ exists, the sequence is called convergent and its limit point is denoted by β^*. A sequence $\left(\beta^k\right)_{k \geq 0}$ is monotonic decreasing if $\beta^{k+1} \subseteq \beta^k$ for all $k \geq 0$.

Lemma 2. Let $\mathcal{T}$ be any model. $\left(\beta^k\right)_{k \geq 0}$ is a monotonic decreasing sequence converging in $\mathcal{B}$.

Proof. Let $\mathcal{T}$ be any model. Let $\left(\beta^k\right)_{k \geq 0}$ be given. By definition $\left(\beta^k\right)_{k \geq 0}$, it holds that $\beta^t \subseteq \beta^k$ for all $k \geq 0$. Thus, $\beta^k \in \mathcal{B}$ for all $k \geq 0$. Therefore, to show that the limit point of $\left(\beta^k\right)_{k \geq 0}$ exists, it suffices to show that $\beta^{k+1} \subseteq \beta^k$ for all $k \geq 0$. To this end, let us proceed by induction. Let $k = 0$. By (4), we see that $\beta^1 \subseteq \beta^0$. Suppose that $\beta^{k+1} \subseteq \beta^k$ for some $k \geq 0$. Let us show that $\beta^{k+2} \subseteq \beta^{k+1}$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Fix any $\hat{t}_i \in \beta_i^{k+2}(t_i)$. (4) implies that $\hat{t}_i \in \beta_i^{k+1}(t_i)$ and that there exists $\nu_i \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}^{k+1}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ such that the inequality in (4) is satisfied for all $y_i \in Y_i^f$.

Since $\nu_i \in \Delta^{\beta_{-i}^{k+1}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, it follows that for all $\left(\theta, t_{-i}, \hat{t}_{-i}\right) \in \Theta \times T_{-i} \times \hat{T}_{-i}$, $\nu_i\left[\theta, t_{-i}, \hat{t}_{-i}\right] > 0 \implies \hat{t}_{-i} \in \beta_{-i}^{k+1}(t_{-i})$. By the inductive hypothesis, it holds that $\left(\theta, t_{-i}, \hat{t}_{-i}\right) \in \Theta \times T_{-i} \times \hat{T}_{-i}$, $\nu_i\left[\theta, t_{-i}, \hat{t}_{-i}\right] > 0 \implies \hat{t}_{-i} \in \beta_{-i}^{k}(t_{-i})$. Moreover, since $\hat{t}_i \in \beta_i^{k+1}(t_i)$, the inductive hypothesis implies that $\hat{t}_i \in \beta_i^k(t_i)$. Thus, we have established that $\hat{t}_i \in \beta_i^{k+1}(t_i)$ and that there exists $\nu_i \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}^k}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ such that the inequality in (4) is satisfied for all $y_i \in Y_i^f$. This implies that $\hat{t}_i \in \beta_i^{k+1}(t_i)$. Since the above arguments hold for all $i \in \mathcal{I}$ and all

$t_i \in T_i$, we have that $\beta^{k+2} \subseteq \beta^{k+1}$. The principle of mathematical induction implies that $\beta^{k+1} \subseteq \beta^k$ for all $k \geq 0$. $\square$

Our condition can be stated as follows.

Definition 6. $f : T \to \Delta(A)$ satisfies *Interim Iterative Monotonicity* (IIM, henceforth) on $\mathcal{T}$ if β^* is an acceptable deception on $\mathcal{T}$ for f.

Theorem 1. If $f : T \to \Delta(A)$ is ICR-implementable on $\mathcal{T}$ by a mechanism satisfying the EBRP, then it satisfies IIM on $\mathcal{T}$.

Proof. Let $\mathcal{T}$ be any model. Let $f : T \to \Delta(A)$ be any SCF. Assume that $\mathcal{M}$ satisfies the EBRP and it ICR-implements f. Lemma 1 implies that there exists a pure strategy $\sigma \in BNE(U(\mathcal{M}, \mathcal{T}))$. This implies that for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$\begin{array}{lr} {}_{(\theta,t_{-i})\in\Theta\times T_{-i}}\kappa(t_i)[\theta, t_{-i}]\, u_i(g(\sigma(t)), \theta) & \geq \\ {}_{(\theta,t_{-i})\in\Theta\times T_{-i}}\kappa(t_i)[\theta, t_{-i}]\, u_i(g((m_i, \sigma_{-i}(t_{-i}))), \theta) & \end{array}$$

for all $m_i \in M_i$. Since $\mathcal{M}$ ICR-implements f, it follows that for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$\begin{array}{lr} {}_{(\theta,t_{-i})\in\Theta\times T_{-i}}\kappa(t_i)[\theta, t_{-i}]\, u_i(f(t), \theta) & \geq \\ {}_{(\theta,t_{-i})\in\Theta\times T_{-i}}\kappa(t_i)[\theta, t_{-i}]\, u_i(g((m_i, \sigma_{-i}(t_{-i}))), \theta) & \end{array} \tag{5}$$

for all $m_i \in M_i$.

Lemma 2 implies that the sequence $\left(\beta^k\right)_{k\geq 0}$ converges to $\beta^* \in \mathcal{B}$. Fix any $t, t' \in T$. Assume that $t' \in \beta^*(t)$. Assume, to the contrary, that $f(t) \neq f(t')$. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, $\Sigma_i(t_i) = \{\sigma_i(t_i') \in M_i | t_i' \in \beta_i^*(t_i)\}$. Then, Σ_i is a correspondence from T_i to $2^{M_i} \backslash \{\emptyset\}$, and so $\Sigma_i \in \mathfrak{S}_i^{\mathcal{M},\mathcal{T}}$. Since $\mathcal{M}$ ICR-implements f, it follows that $\Sigma \in \mathfrak{S}^{\mathcal{M},\mathcal{T}}$ cannot be a best-reply set in $U(\mathcal{M}, \mathcal{T})$. Then, for some $\left(i, t_i, \sigma\left(\hat{t}_i\right)\right) \in \mathcal{I} \times T_i \times \Sigma_i(t_i)$ and all $\pi_i(t_i) \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$, it holds that

$$\sigma_i\left(\hat{t}_i\right) \notin BR_i\left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)\,|\mathcal{M}\right),$$

and so

$$\begin{array}{lr} \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\text{marg}_{\Theta\times M_{-i}}\pi_i\left(t_i\right)\left[\theta,m_{-i}\right]\right)\left[u_i\left(g\left(m_i,m_{-i}\right),\theta\right)\right] & > \\ \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\text{marg}_{\Theta\times M_{-i}}\pi_i\left(t_i\right)\left[\theta,m_{-i}\right]\right)\left[u_i\left(g\left(\sigma_i\left(\hat{t}_i\right),m_{-i}\right),\theta\right)\right] & \end{array} \tag{6}$$

for some $m_i \in M_i$.

For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta^*_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be any distribution. For all $i \in \mathcal{I}$, all $t_i \in T_i$, let $\bar{\pi}_i(t_i) \in \Delta(\Theta \times T_{-i} \times M_{-i})$ be defined, for all $(\theta, t_{-i}, m_{-i}) \in \Theta \times T_{-i} \times M_{-i}$, by

$$\bar{\pi}_i(t_i)[\theta, t_{-i}, m_{-i}] = \sum_{\hat{t}_{-i}\in\sigma_{-i}^{-1}(m_{-i})} \nu_i(t_i)\left[\theta, t_{-i}, \hat{t}_{-i}\right],$$

where $\sigma_{-i}^{-1}(m_{-i}) = \Pi_{j\in\mathcal{I}\setminus\{i\}}\sigma_j^{-1}(m_j)$ and $\sigma_j^{-1}(m_j) = \{t_j \in T_j | m_j = \sigma_j(t_j)\}$. Since $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, we have that $\text{marg}_{\Theta\times T_{-i}}\nu_i(t_i) = \kappa(t_i)$. Moreover, by construction, $\text{marg}_{\Theta\times T_{-i}}\nu_i(t_i) = \text{marg}_{\Theta\times T_{-i}}\bar{\pi}_i(t_i)$.[6] Moreover, since $\nu_i(t_i) \in \Delta^{\beta^*_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, it also follows that for all $(\theta, t_{-i}, m_{-i}) \in \Theta \times T_{-i} \times M_{-i}$, $\bar{\pi}_i(t_i)[\theta, t_{-i}, m_{-i}] > 0 \implies m_{-i} \in \Sigma_{-i}(t_{-i})$. Thus, we have that $\bar{\pi}_i(t_i) \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Moreover, by construction, we also have that for all $i \in \mathcal{I}$

[6] Observe that for all $(\theta, t_{-i}) \in \Theta \times T_{-i}$,

$$\begin{aligned} \text{marg}_{\Theta\times T_{-i}}\bar{\pi}_i(t_i)[\theta, t_{-i}] &= \sum_{m_{-i}\in M_{-i}} \bar{\pi}_i(t_i)[\theta, t_{-i}, m_{-i}] \\ &= \sum_{m_{-i}\in M_{-i}} \left(\sum_{\hat{t}_{-i}\in\sigma_{-i}^{-1}(m_{-i})} \nu_i(t_i)\left[\theta, t_{-i}, \hat{t}_{-i}\right]\right) \\ &= \sum_{\hat{t}_{-i}\in\hat{T}_{-i}} \nu_i(t_i)\left[\theta, t_{-i}, \hat{t}_{-i}\right] \\ &= \text{marg}_{\Theta\times T_{-i}}\nu_i(t_i)[\theta, t_{-i}]. \end{aligned}$$

and all $m_i \in M_i$,[7]

$$\begin{array}{ll} \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\mathrm{marg}_{\Theta\times M_{-i}}\bar{\pi}_i(t_i)[\theta,m_{-i}]\right) u_i(g(m_i,m_{-i}),\theta) & = \\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right). & \end{array} \tag{7}$$

Since $\bar{\pi}_i(t_i) \in \Delta^{\kappa(t_i)}(\Theta\times T_{-i}\times M_{-i}) \cap \Delta^{\Sigma_{-i}}(\Theta\times T_{-i}\times M_{-i})$ for all $i\in\mathcal{I}$ and all $t_i\in T_i$, from (6) and (7), we have that for some $\left(i,t_i,\sigma\left(\hat{t}_i\right)\right)\in\mathcal{I}\times T_i\times\Sigma_i(t_i)$,

$$\begin{array}{ll} \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right) & > \\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(\sigma_i\left(\hat{t}_i\right),\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right). & \end{array} \tag{8}$$

Since $\mathcal{M}$ ICR-implements f, it also follows that $g\left(\sigma_i\left(\hat{t}_i\right),\sigma_{-i}\left(\hat{t}_{-i}\right)\right) = f\left(\hat{t}_i,\hat{t}_{-i}\right)$. Since $\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right] = \kappa(t_i)$, we have that (8) is equivalent to

$$\begin{array}{ll} \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \kappa(t_i)\left[\theta,\hat{t}_{-i}\right] u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right) & > \\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \kappa(t_i)\left[\theta,\hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right), & \end{array}$$

which contradicts (5). Thus, f satisfies IIM. □

Theorem 2. $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$ if and only if f is IRM on $\mathcal{T}$.

Proof. Assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$. Take any unacceptable

[7] To see it, observe that

$$\begin{aligned} & \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\mathrm{marg}_{\Theta\times M_{-i}}\bar{\pi}_i(t_i)[\theta,m_{-i}]\right) u_i(g(m_i,m_{-i}),\theta) \\ = & \sum_{(\theta,t_{-i},m_{-i})\in\Theta\times T_{-i}\times M_{-i}} \bar{\pi}_i(t_i)[\theta,t_{-i},m_{-i}]\, u_i(g(m_i,m_{-i}),\theta) \\ = & \sum_{(\theta,t_{-i},m_{-i})\in\Theta\times T_{-i}\times M_{-i}} \left(\sum_{\hat{t}_{-i}\in\sigma_{-i}^{-1}(m_{-i})} \nu_i(t_i)\left[\theta,t_{-i},\hat{t}_{-i}\right] u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right)\right) \\ = & \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\sum_{\hat{t}_{-i}\in\sigma_{-i}^{-1}(m_{-i})} \left(\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right)\right) \\ = & \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right). \end{aligned}$$

deception profile $\beta \in \mathcal{B}$ on $\mathcal{T}$ for f. Assume, to the contrary, that for all $(i, t_i, t_i') \in \mathcal{I} \times T_i \times \beta_i(t_i)$, there exists $\psi_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ such that for all $y_i^* \in Y_i^f$, (3) is violated.[8] To derive a contradiction, let us first show that $\beta \subseteq \beta^k$ for all $k \geq 0$. Let us proceed by induction.

Since $\beta \subseteq \bar{\beta} = \beta^0$, by definition, let us suppose that for some $k \geq 0$, it holds that it follows that $\beta \subseteq \beta^k$. Let us show that $\beta \subseteq \beta^{k+1}$. By the inductive hypothesis, it holds that $\psi_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta^k_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Take any $\hat{t}_i \in \beta_i(t_i)$. It follows from the inductive hypothesis that $\hat{t}_i \in \beta_i^k(t_i)$. Since (3) is violated for $y_i^* \in Y_i^f$, (4) implies that $\hat{t}_i \in \beta_i^{k+1}(t_i)$. Since the triplet $\left(i, t_i, \hat{t}_i\right) \in \mathcal{I} \times T_i \times \beta_i(t_i)$ was chosen arbitrarily, we conclude that $\beta \subseteq \beta^{k+1}$. By the principle of mathematical induction, it holds that $\beta \subseteq \beta^k$ for all $k \geq 0$. Since $\left(\beta^k\right)_{k\geq 0}$ converges to $\beta^* \in \mathcal{B}$, and from the proof of Lemma 2 we know that $\beta^{k+1} \subseteq \beta^k$ for all $k \geq 0$, we have that $\beta \subseteq \beta^*$. Since f satisfies IIM on $\mathcal{T}$, it follows that β^* is an acceptable deception profile on $\mathcal{T}$ for f, and so β is also an acceptable deception profile on T for f, which is a contradiction.

Assume f is IRM on $\mathcal{T}$. Assume, to the contrary, that $\beta^* \in \mathcal{B}$ is not acceptable on $\mathcal{T}$ for f. Since f is IRM, it follows that there exists $(i, t_i, t_i') \in \mathcal{I} \times T_i \times \beta_i^*(t_i)$ such that for all $\psi_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta^*_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, there exists $y_i^* \in Y_i^f$ such that (3) is satisfied. Recall that in the proof of Lemma 2 we have shown that $\beta^{k+1} \subseteq \beta^k$ for all $k \geq 0$ and that β^* is the limit point of $\left(\beta^k\right)_{k\geq 0}$. Since $t_i' \in \beta_i^*(t_i)$, (4) implies that there exists $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta^*_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ such that

$$
\begin{array}{l}
\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \left(\text{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i)\left[\theta, \hat{t}_{-i}\right]\right) u_i\left(f\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) \quad \geq \\
\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \left(\text{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i)\left[\theta, \hat{t}_{-i}\right]\right) u_i\left(y_i^*\left(\hat{t}_{-i}\right), \theta\right)
\end{array}
$$

for all $y_i^* \in Y_i^f$, yielding a contradiction. □

Any SCF satisfying our condition on $\mathcal{T}$ is *incentive compatible* on $\mathcal{T}$. The condition can be stated as follows.

[8]Recall that Y^f is a nonempty metrizable subspace.

Definition 7. $f : T \to \Delta(A)$ *incentive compatible* on $\mathcal{T}$ if for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$\sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta,t_{-i}]\, u_i(f(t_i,t_{-i}),\theta) \geq \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta,t_{-i}]\, u_i(f(t_i',t_{-i}),\theta)$$

for all $t_i \in T_i$.

Theorem 3. $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$ implies that f is incentive compatible on $\mathcal{T}$.

Proof. It follows from Theorem 2 and Lemma 3 of Oury and Tercieux (2012). □

IV. A full characterization

Before stating and proving our characterization result, let us briefly discuss why our condition is sufficient for f to be ICR-implementable on $\mathcal{T}$. To this end, we need additional notation. Let $\mathcal{T}$ be any model. Fix any $\beta \in \mathcal{B}$, and any $i \in \mathcal{I}$. Let $\Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$ be defined by

$$\Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right) = \left\{ \psi_i \;\middle|\; \begin{array}{l} \text{There exists } \nu_i(t_i) \in \Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \\ \text{such that } \operatorname{marg}_{\Theta\times\hat{T}_{-i}} \nu_i(t_i) = \psi_i. \end{array} \right\} \tag{9}$$

Since for all $t_{-i} \in T_{-i}$, $\bar{\beta}_{-i}(t_{-i}) = T_{-i}$, it follows that $\Delta^{\bar{\beta}_{-i}}\left(\Theta \times \hat{T}_{-i}\right) = \Delta\left(\Theta \times \hat{T}_{-i}\right)$.

The following definition is critical in the construction of our implementing mechanism.

Definition 8. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$ and all $i \in \mathcal{I}$, $i \in \mathcal{I}(\beta)$ if and only if for all $\psi_i \in \Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exist $y_i, \bar{y}_i \in Y_i^f$ such that

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(y_i\left(\hat{t}_{-i}\right),\theta\right) > \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right). \tag{10}$$

The above definition says that $i \in \mathcal{I}(\beta)$ provided that for every belief ψ_i of player i over $\Theta \times \hat{T}_{-i}$, there are mappings $y_i, \bar{y}_i \in Y_i^f$ that may depend on his belief ψ_i such that y_i is strictly preferred to $\bar{y}_i$, given his belief ψ_i. A stronger, though more desirable, definition would be to require that the mapping $\bar{y}_i$ does not depend on player i's belief. The definition can be stated as follows.

Definition 9. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$ and all $i \in \mathcal{I}$, $i \in \mathcal{I}^*(\beta)$ if and only if there exists $\bar{y}_i \in Y_i^f$ such that for all $\psi_i \in \Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exists $y_i \in Y_i^f$ such that

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(y_i\left(\hat{t}_{-i}\right),\theta\right) > \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right). \quad (11)$$

Observe that Definition 9 is equivalent to Assumption 1 of Oury and Tercieux (2012) when $\beta = \bar{\beta}$. We show below that Definition 8 and Definition 9 are equivalent.

Lemma 3. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$, $\mathcal{I}^*(\beta) = \mathcal{I}(\beta)$.

Proof. Let $\mathcal{T}$ be any model. Fix any $\beta \in \mathcal{B}$. Since it is clear that $\mathcal{I}^*(\beta) \subseteq \mathcal{I}(\beta)$, let us show that $\mathcal{I}(\beta) \subseteq \mathcal{I}^*(\beta)$. Assume that $i \in \mathcal{I}(\beta)$. Definition 8 implies that for all $\psi_i \in \Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exist $y_i^{\psi_i}, \bar{y}_i^{\psi_i} \in Y_i^f$ such that (11) is satisfied. Since $\Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$ is a separable metric space, let $\hat{\Delta}\left(\Theta \times \hat{T}_{-i}\right) = \cup_{k\in\mathbb{N}}\{\psi_{i,k}\}$ be a countable, dense subset of $\Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$. Let $\tilde{y}_i \in Y_i^f$ be a mapping defined by

$$\tilde{y}_i = \sum_{k=1}^{\infty} \frac{1}{2^k} \bar{y}_i^{\psi_{i,k}}.$$

For all $\bar{k} \in \mathbb{N}$, let $y_i^{\psi_{i,\bar{k}}} \in Y_i^f$ be a mapping defined by

$$y_i^{\bar{k}} = \sum_{k\neq\bar{k}} \frac{1}{2^k} \bar{y}_i^{\psi_{i,k}} + \frac{1}{2^{\bar{k}}} y_i^{\psi_{i,\bar{k}}}.$$

Thus, for all $k \in \mathbb{N}$, we have that

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_{i,k}\left[\theta,\hat{t}_{-i}\right] u_i\left(y_i^k\left(\hat{t}_{-i}\right),\theta\right) > \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\tilde{y}_i\left(\hat{t}_{-i}\right),\theta\right),$$

where the strict inequality is guaranteed by (10). Since player i's preference over lotteries are continuous and since, moreover, $\hat{\Delta}\left(\Theta \times \hat{T}_{-i}\right)$ is a countable, dense subset of $\Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, it follows that $i \in \mathcal{I}^*(\beta)$. Since the choice of $i \in \mathcal{I}(\beta)$ was arbitrary, it follows that $\mathcal{I}(\beta) \subseteq \mathcal{I}^*(\beta)$. □

In what follows, to avoid trivialities, we assume that $\mathcal{I}\left(\bar{\beta}\right) \neq \emptyset$. Moreover, we will also assume that $\mathcal{I}(\beta^*) = \mathcal{I}$. The reason is that if $\mathcal{I}(\beta^*) \neq \mathcal{I}$, part (ii) of the above lemma implies that the planner's objective is constant on $\Pi_{i \in \mathcal{I}^c(\beta^*)} T_i \equiv T_{\mathcal{I}^c(\beta^*)}$, where $\mathcal{I}^c(\beta^*)$ is the complement of $\mathcal{I}^c(\beta^*)$. Therefore, the planner can, equivalently, focus on the implementation of an SCF $\hat{f} : \Pi_{i \in \mathcal{I}(\beta^*)} T_i \rightarrow \Delta(A)$ defined, for all $t \in \Pi_{i \in \mathcal{I}(\beta^*)} T_i$, by $\hat{f}(t) = f(t, t')$ for all $t' \in T_{\mathcal{I}^c(\beta^*)}$. This is justified by the following lemmata.

Lemma 4. Assume that $f : T \rightarrow \Delta(A)$ satisfies IIM on $\mathcal{T}$. For all $k \geq 0$ and all $i \in \mathcal{I}$, $i \in \mathcal{I}^c\left(\beta^k\right) \implies \beta_i^{k+1} = \beta_i^k = \bar{\beta}_i$.

Proof. Assume that $f : T \rightarrow \Delta(A)$ satisfies IIM on $\mathcal{T}$. Fix any $k \geq 0$. Assume that $i \in \mathcal{I}^c\left(\beta^k\right)$. Assume, to the contrary, $\beta_i^{k+1} \neq \beta_i^k$. Since Lemma 2 implies that $\left(\beta^k\right)_{k \geq 0}$ is a monotonic decreasing sequence, it follows that there exists $\left(t_i, \hat{t}_i\right)$ such that $\hat{t}_i \in \beta_i^k(t_i)$ and $\hat{t}_i \notin \beta_i^{k+1}(t_i)$. It follows from (4) that for all $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}^k}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$,

$$\begin{array}{l} \sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \left(\operatorname{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i)\left[\theta, \hat{t}_{-i}\right]\right) u_i\left(f\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) \quad < \\ \sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \left(\operatorname{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i)\left[\theta, \hat{t}_{-i}\right]\right) u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right), \theta\right) \end{array}$$

for some $\bar{y}_i \in Y_i^f$. Therefore, for all $\psi_i \in \Delta^{\beta_{-i}^k}\left(\Theta \times \hat{T}_{-i}\right)$,

$$\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \psi_i\left[\theta, \hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) < \sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \psi_i\left[\theta, \hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right), \theta\right)$$

for some $\bar{y}_i \in Y_i^f$. Let $y_i\left(\hat{t}_i, \cdot\right) = f\left(\hat{t}_i, \cdot\right)$. Since f satisfies IIM on $\mathcal{T}$, Theorem 2 and Theorem 3 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i\left(\hat{t}_i, \cdot\right) \in Y_i^f$. Definition 8 implies that $i \in \mathcal{I}\left(\beta^k\right)$, yielding a contradiction.

Finally, let us show that $\beta_i^{k+1} = \beta_i^k = \bar{\beta}_i$. Assume, to the contrary, that $\beta_i^{k+1} = \beta_i^k \neq \bar{\beta}_i$. Since Lemma 2 implies that $\left(\beta_i^k\right)_{k \geq 0}$ is a decreasing monotonic sequence, it follows that there exists $\hat{k}$ such that $0<\hat{k}\leq k$ such that $\beta_i^{\hat{k}}\subseteq\beta_i^{\hat{k}-1}$ and $\beta_i^{\hat{k}}\neq\beta_i^{\hat{k}-1}$. It follows that $\beta_i^{\hat{k}}(t_i)\subseteq\beta_i^{\hat{k}-1}(t_i)$ and $\beta_i^{\hat{k}}(t_i)\neq\beta_i^{\hat{k}-1}(t_i)$ for some $t_i\in T_i$, and so $\hat{t}_i\in\beta_i^{\hat{k}-1}(t_i)$ and $t_i\notin\beta_i^{\hat{k}}(t_i)$ for some $\hat{t}_i,t_i\in T_i$. (4) implies that there exists $\bar{y}_i\in Y_i^f$ such that

$$\begin{array}{l}\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)\quad<\\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)\end{array}$$

for all $\nu_i(t_i)\in\Delta^{\kappa(t_i)}\left(\Theta\times T_{-i}\times\hat{T}_{-i}\right)\cap\Delta^{\beta_{-i}^{\hat{k}-1}}\left(\Theta\times T_{-i}\times\hat{T}_{-i}\right)$. By definition of $\Delta^{\beta_{-i}^{\hat{k}-1}}\left(\Theta\times\hat{T}_{-i}\right)$ in (9), it follows that there exists $\bar{y}_i\in Y_i^f$ such that

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\psi_i\left[\theta,\hat{t}_{-i}\right]u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)<\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\psi_i\left[\theta,\hat{t}_{-i}\right]u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i\in\Delta^{\beta_{-i}^{\hat{k}-1}}\left(\Theta\times\hat{T}_{-i}\right)$. Let $y_i\left(\hat{t}_i,\cdot\right)=f\left(\hat{t}_i,\cdot\right)$. Since f satisfies IIM on $\mathcal{T}$, Theorem 2 and Theorem 3 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i\left(\hat{t}_i,\cdot\right)\in Y_i^f$. Definition 8 implies that $i\in\mathcal{I}\left(\beta^{\hat{k}-1}\right)$. Since Lemma 2 implies that $\left(\beta_i^k\right)_{k\geq 0}$ is a decreasing monotonic sequence and since, moreover, $\hat{k}$ is such that $0<\hat{k}\leq k$, it follows that there exist $\bar{y}_i,y_i\left(\hat{t}_i,\cdot\right)\in Y_i^f$ such that

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\psi_i\left[\theta,\hat{t}_{-i}\right]u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)<\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\psi_i\left[\theta,\hat{t}_{-i}\right]u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i\in\Delta^{\beta_{-i}^{k}}\left(\Theta\times\hat{T}_{-i}\right)\subseteq\Delta^{\beta_{-i}^{\hat{k}-1}}\left(\Theta\times\hat{T}_{-i}\right)$. Defintion 8 implies that $i\in\mathcal{I}\left(\beta^k\right)$, which is a contradiction. Thus, $\beta_i^{k+1}=\beta_i^k=\bar{\beta}_i$. $\square$

Lemma 5. Assume that $f:T\to\Delta(A)$ satisfies IIM on $\mathcal{T}$.

(i) If $\mathcal{I}\left(\bar{\beta}\right)=\emptyset$, then f is constant.[9]

(ii) If $\mathcal{I}\left(\beta^*\right)\neq\mathcal{I}$, then for all $i\in\mathcal{I}^c\left(\beta^*\right)$, all $t_{-i}\in T_{-i}$ and all $t_i,t_i'\in T_i$, $f\left(t_i,t_{-i}\right)=f\left(t_i',t_{-i}\right)$.

[9] f is constant if for all $t,t'\in T$, $f(t)=f(t')$.

Proof. Assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$. To show part (i), assume that $\mathcal{I}\left(\bar{\beta}\right) = \emptyset$. If $\beta^* = \bar{\beta}$, then $\bar{\beta}$ is an acceptable deception profile on $\mathcal{T}$ for f. This implies that f is constant. Thus, the complete the proof, let us show that $\beta^* = \bar{\beta}$. Assume, to the contrary, that $\beta^* \neq \bar{\beta}$. Then, there exists $(i, t_i) \in \mathcal{I} \times T_i$ such that $\beta_i^*(t_i) \neq T_i = \bar{\beta}_i(t_i)$. Since $\beta_i^*(t_i) \subseteq \bar{\beta}_i(t_i) = T_i$, it follows that there exists $\hat{t}_i \in \bar{\beta}_i(t_i) = T_i$ such that $\hat{t}_i \notin \beta_i^*(t_i)$. Since β^* is the limit point of $\left(\beta^k\right)_{k\geq 0}$ and since, by Lemma 2, $\beta^* \subseteq \beta^k$ for all $k \geq 0$, it follows from (4) and the fact that $\beta_i^0(t_i) = \bar{\beta}_i(t_i)$ that there exists $k+1$ such that $\hat{t}_i \notin \beta_i^{k+1}(t_i)$, $\hat{t}_i \in \beta_i^k(t_i)$ and for all $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}^k}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$,

$$\begin{array}{l}\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \quad < \\ \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)\end{array}$$

for some $\bar{y}_i \in Y_i^f$. Since $\Delta^{\beta_{-i}^k}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \subseteq \Delta^{\bar{\beta}_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) = \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, we can write that for all $\psi_i \in \Delta\left(\Theta \times \hat{T}_{-i}\right)$,

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for some $\bar{y}_i \in Y_i^f$. Let $y_i\left(\hat{t}_i, \cdot\right) = f\left(\hat{t}_i, \cdot\right)$. Since f satisfies IIM on $\mathcal{T}$, Theorem 2 and Theorem 3 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i\left(\hat{t}_i, \cdot\right) \in Y_i^f$. Definition 8 implies that $i \in \mathcal{I}\left(\bar{\beta}\right)$, yielding a contradiction. This completes the proof of part (i).

Let us show part (ii). Assume that $\mathcal{I}(\beta^*) \neq \mathcal{I}$. Suppose that $\beta_i^* = \bar{\beta}_i$ for all $i \in \mathcal{I}^c(\beta^*)$. Since f satisfies IIM on $\mathcal{T}$, it follows that β^* is an acceptable deception profile on $\mathcal{T}$ for f. Fix any $i \in \mathcal{I}^c(\beta^*)$ and any $t_i \in T_i$. Since $\beta_i^* = \bar{\beta}_i$, we have that $\beta_i^*(t_i) = \bar{\beta}_i(t_i) = T_i$. Since f satisfies IIM on $\mathcal{T}$, we have that for all $t_{-i} \in T_{-i}$, $f(t_i', t_{-i}) = f(t_i'', t_{-i})$ for all $t_i', t_i'' \in \beta_i^*(t_i) = \bar{\beta}_i(t_i) = T_i$. Since the choice of $i \in \mathcal{I}^c(\beta^*)$ was arbitrary, the statement of part (ii) follows if we show that $\beta_i^* = \bar{\beta}_i$ for all $i \in \mathcal{I}^c(\beta^*)$. To this end, fix any $i \in \mathcal{I}^c(\beta^*)$. Assume that $\beta_i^* \neq \bar{\beta}_i$. Then, there exists $t_i \in T_i$ such that $\beta_i^*(t_i) \neq T_i = \bar{\beta}_i(t_i)$. A contradiction can be derived by using the same reasoning used in part (i). This completes the proof of part (ii). $\square$

The following result is useful in defining *Rule 3* of the mechanism.

Lemma 6. Let $\mathcal{T}$ be any model. For all $i \in \mathcal{I}(\beta^*)$, there exists $\hat{y}_i \in \Delta(A)$ such that for all $\phi_i \in \Delta(\Theta)$, there exists $y_i \in \Delta(A)$ such that

$$\sum_{\theta \in \Theta} \phi_i(\theta) u_i(y_i, \theta) > \sum_{\theta \in \Theta} \phi_i(\theta) u_i(\hat{y}_i, \theta). \tag{12}$$

Proof. Fix any $i \in \mathcal{I}(\beta^*)$. Lemma 3 implies that $i \in \mathcal{I}^*(\beta^*)$. Definition 9 implies that there exists $\bar{y}_i \in Y_i^f$ such that for all $\psi_i \in \Delta^{\beta^*_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exists $y_i \in Y_i^f$ such that (11) holds. Since $\beta^t \subseteq \beta^*$, it follows that there exists $\bar{y}_i \in Y_i^f$ such that for all $\psi_i \in \Delta^{\beta^t_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exists $y_i \in Y_i^f$ such that (11) holds. Fix any $t_i \in T_i$. Observe that $\phi_i \circ \left(\text{marg}_{T_{-i}} \kappa(t_i)\right) \in \Delta^{\beta^t_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$ for all $\phi_i \in \Delta(\Theta)$. Therefore, it holds that

$$\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \left(\phi_i[\theta]\left(\text{marg}_{T_{-i}} \kappa(t_i)\left[\hat{t}_{-i}\right]\right)\right)\left[u_i\left(y_i\left(\hat{t}_{-i}\right), \theta\right) - u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right), \theta\right)\right] > 0.$$

By setting

$$y_i = \sum_{\hat{t}_{-i} \in \hat{T}_{-i}} \left(\text{marg}_{T_{-i}} \kappa(t_i)\left[\hat{t}_{-i}\right]\right) y_i\left(\hat{t}_{-i}\right)$$

and

$$\hat{y}_i = \sum_{\hat{t}_{-i} \in \hat{T}_{-i}} \left(\text{marg}_{T_{-i}} \kappa(t_i)\left[\hat{t}_{-i}\right]\right) \bar{y}_i\left(\hat{t}_{-i}\right),$$

and by noting that $y_i, \hat{y}_i \in \Delta(A)$, the inequality in (12) follows for i. Since the choice of $i \in \mathcal{I}(\beta^*)$ was arbitrary, the statement follows. $\square$

Let $\mathcal{T}$ be any model. Since $\mathcal{I}(\beta^*) = \mathcal{I}$ and since Lemma 6 guarantees the existence of the lottery $\hat{y}_i \in \Delta(A)$ for all $i \in \mathcal{I}$, let us define the lottery $\hat{y}$ by

$$\hat{y} = \frac{1}{I} \sum_{i \in \mathcal{I}} \hat{y}_i.$$

Given the sequence $\left(\beta^k\right)_{k\geq 0}$, for every $i \in \mathcal{I}$, let $k(i)$ be the lowest integer such that $i \in \mathcal{I}^*\left(\beta^{k(i)}\right) \backslash \mathcal{I}^*\left(\beta^{k(i)-1}\right)$. For player $i \in \mathcal{I}^*\left(\beta^{k(i)}\right) \backslash \mathcal{I}^*\left(\beta^{k(i)-1}\right)$, Definition 11 implies that there exists $\bar{y}_i \in Y_i^f$ satisfying (11). Let us denote $\bar{y}_i$ by $\bar{y}_i^{\beta^{k(i)}}$. Since $\bar{y}_i^{\beta^{k(i)}} \in Y_{i,s}^f$, we can choose an $\varepsilon > 0$ sufficiently small such that the mapping $\mathfrak{y}_i^{\beta^{k(i)}} : T_{-i} \to \Delta(A)$ defined by

$$\mathfrak{y}_i^{\beta^{k(i)}}(t_{-i}) = (1-\varepsilon)\,\bar{y}_i^{\beta^{k(i)}}(t_{-i}) + \varepsilon \hat{y} \tag{13}$$

is such that $\mathfrak{y}_i^{\beta^{k(i)}} \in Y_{i,s}^f$.

Let us now define the mechanism $\mathcal{M}$. For all $i \in \mathcal{I}$, let

$$M_i = M_i^1 \times M_i^2 \times M_i^3 \times M_i^4,$$

where

$$M_i^1 = T_i,\ M_i^2 = \mathbb{N},\ M_i^3 = Y_i^* \text{ and } M_i^4 = \Delta^*(A),$$

where $\mathbb{N}$ is the set of natural numbers, Y_i^* is a countable, dense subset of Y_i^f, and $\Delta^*(A)$ is a countable, dense subset of $\Delta(A)$. For all $m \in M$, let $g : M \to \Delta(A)$ be defined as follows.

Rule 1: If $m_i^2 = 1$ for all $i \in \mathcal{I}$, then $g(m) = f(m^1)$.

Rule 2: For all $i \in \mathcal{I}$, if $m_j^2 = 1$ for all $j \in \mathcal{I}\backslash\{i\}$ and $m_i^2 > 1$, then

$$g(m) = m_i^3\left(m_{-i}^1\right)\left(1 - \frac{1}{1+m_i^2}\right) \oplus \mathfrak{y}_i^{\beta^{k(i)}}\left(m_{-i}^1\right)\left(\frac{1}{1+m_i^2}\right), \tag{14}$$

where $\mathfrak{y}_i^{\beta^{k(i)}} \in Y_{i,s}^f$ is defined in (13).

Rule 3: Otherwise, for each $i \in \mathcal{I}$, m_i^4 is picked with probability $\frac{1}{I}\left(1 - \frac{1}{1+m_i^2}\right)$ and $\hat{y}_i$ is picked with probability $\frac{1}{I}\left(\frac{1}{1+m_i^2}\right)$; that is,

$$g(m) = \frac{1}{I}\left[m_i^4\left(1 - \frac{1}{1+m_i^2}\right) \oplus \hat{y}_i\left(\frac{1}{1+m_i^2}\right)\right], \tag{15}$$

where $\hat{y}_i$ is specified by Lemma 6.

Suppose that f satisfies IIM on $\mathcal{T}$. In what follows, we prove that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$ and that $\mathcal{M}$ satisfies the EBRP. The following lemmata will help us to complete the proof.

Lemma 7. *$BNE\left(U\left(\mathcal{M},\mathcal{T}\right)\right)\neq\emptyset$.*

Proof. For all $i\in\mathcal{I}$, let $\sigma_i:T_i\rightarrow M_i$ be defined by $\sigma_i\left(t_i\right)=\left(t_i,1,\cdot,\cdot\right)$. For all $i\in\mathcal{I}$ and all $t_i\in T_i$, let $\pi_i\left(t_i\right)\in\Delta\left(\Theta\times T_i\times M_{-i}\right)$ be defined by

$$\pi_i\left(t_i\right)\left[\theta,t_i,m_{-i}\right]=\kappa\left(t_i\right)\left[\theta,t_{-i}\right]\delta_{\sigma_{-i}\left(t_{-i}\right)}\left[m_{-i}\right],$$

where $\delta_{\sigma_{-i}\left(t_{-i}\right)}$ is the dirac measure on $\left\{\sigma_{-i}\left(t_{-i}\right)\right\}$. By construction, for all $t_i\in T_i$ and all $\left(\theta,t_{-i},m_{-i}\right)\in\Theta\times T_i\times M_{-i}$, $\pi_i\left(t_i\right)\left[\theta,t_i,m_{-i}\right]>0\implies m_{-i}=\sigma_{-i}\left(t_{-i}\right)$. Moreover, by construction and *Rule 1*, for all $i\in\mathcal{I}$ and all $t_i\in T_i$,

$$\begin{aligned}&\sum_{\left(\theta,m_{-i}\right)\in\Theta\times M_{-i}}\text{marg}_{\Theta\times M_{-i}}\pi_i\left(t_i\right)\left[\theta,m_{-i}\right]u_i\left(g\left(\sigma_i\left(t_i\right),m_{-i}\right),\theta\right)\\=&\sum_{\left(\theta,t_{-i}\right)\in\Theta\times T_{-i}}\kappa\left(t_i\right)\left[\theta,t_{-i}\right]u_i\left(f\left(t_i,t_{-i}\right),\theta\right).\end{aligned}$$

Finally, by definition of g and the fact that f is incentive compatible on $\mathcal{T}$ (Theorem 3), it follows that for all $i\in\mathcal{I}$ and all $t_i\in T_i$, $\text{Supp}(\sigma_i\left(t_i\right))\subseteq BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i|\mathcal{M}\right)$, and so $\sigma\in BNE\left(U\left(\mathcal{M},\mathcal{T}\right)\right)$.

Before proving the following lemma, let us introduce the following definitions. For all $\beta\in\mathcal{B}$ and all $i\in\mathcal{I}$, define $\Sigma_i^{\beta_i}:T_i\rightarrow2^{M_i}\backslash\left\{\emptyset\right\}$ by

$$\Sigma_i^{\beta_i}\left(t_i\right)=\left\{m_i\in M_i|m_i^1\in\beta_i\left(t_i\right)\right\},\tag{16}$$

and define $\tilde{\Sigma}_i^{\beta_i}:T_i\rightarrow2^{M_i}\backslash\left\{\emptyset\right\}$ by

$$\tilde{\Sigma}_i^{\beta_i}\left(t_i\right)=\left\{m_i\in\Sigma_i^{\beta_i}\left(t_i\right)|m_i^2=1\right\}.\tag{17}$$

It can be checked that $\Sigma^\beta,\tilde{\Sigma}^\beta\in\mathfrak{S}^{\mathcal{M},\mathcal{T}}$. □

Lemma 8. For all $k \geq 0$, all $i \in \mathcal{I}\left(\beta^k\right)$ and all $\pi_i \in \Delta\left(\Theta \times T_{-i} \times M_{-i}\right)$, if

$$m_i \in BR_i\left(\text{marg}_{\Theta \times M_{-i}} \pi_i | \mathcal{M}\right) \tag{18}$$

and

$$\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^k}}\left(\Theta \times T_{-i} \times M_{-i}\right),$$

then $m_i^2 = 1$ and

$$\pi_i \in \Delta^{\tilde{\Sigma}_{-i}^{\beta_{-i}^k}}\left(\Theta \times T_{-i} \times M_{-i}\right).$$

Proof. Fix any $k \geq 0$ and any $i \in \mathcal{I}\left(\beta^k\right)$. Suppose that $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^k}}\left(\Theta \times T_{-i} \times M_{-i}\right)$ and that $m_i \in BR_i\left(\text{marg}_{\Theta \times M_{-i}} \pi_i | \mathcal{M}\right)$. Let us first show that $m_i^2 = 0$. Assume, to the contrary, that $m_i^2 > 0$. Let us proceed according to whether *Rule 2* applies or *Rule 3* applies. Before we begin the proof we argue that $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^k}}\left(\Theta \times T_{-i} \times M_{-i}\right)$ implies that

$$\underbrace{\sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^k}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}_{Prob[Rule2]} + \underbrace{\sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^k, c}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}_{Prob[Rule3]} = 1 \tag{19}$$

For every i, t_i, define $\nu_i(t_i) \in \Delta(\Theta \times T_{-i} \times M_{-i}^1)$ as follows:

$$\nu_i(t_i)[\theta, t_{-i}, m_{-i}^1] = \frac{\sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^k}(t_{-i})[m_{-i}^1]} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}{Prob[Rule2]} \tag{20}$$

Notice that since $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^k}}\left(\Theta \times T_{-i} \times M_{-i}\right)$ implies that $\nu_i(t_i) \in \Delta^{\beta_{-i}^k}(\Theta \times T_{-i}^1 \times M_{-i}^1)$. Let $\psi_i = marg_{\Theta \times M_{-i}^1} \nu_i(t_i)$. Since $\nu_i(t_i) \in \Delta^{\beta_{-i}^k}(\Theta \times T_{-i}^1 \times M_{-i}^1)$, it holds that

$$\psi_i \in \Delta^{\beta^{k-1}}(\Theta \times M_{-i}^1) \tag{21}$$

Define $\phi_i(\theta) \in \Delta(\Theta)$ as follows:

$$\phi_i(\theta) = \frac{\sum_{t_{-i}\in T_{-i}} \sum_{m_{-i}\in\tilde{\Sigma}_{-i}^{\beta^k_{-i},c}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}{Prob[Rule3]} \tag{22}$$

The utility of m_i under the beliefs $marg_{\Theta\times M_{-i}}\pi_i$ denoted by $U_i(m_i,\pi_i)$ is given by

$$\begin{aligned} U_i(m_i,\pi_i) = \alpha\sum_{\theta,t_{-i}}\psi_i(\theta,t_{-i})u_i\Big[\big(1-\frac{1}{m_i^2+1})m_i^3(t_{-i})\oplus\frac{1}{m_i^2+1}\mathfrak{y}_i^{\beta_i^k}(t_{-i})\Big),\theta\Big] \\ +(1-\alpha)\sum_{\theta}\phi_i(\theta)u_i\Big[\big(1-\frac{1}{m_i^2+1})m_i^4\oplus\frac{1}{m_i^2+1}\hat{y}_i\Big),\theta\Big] \end{aligned} \tag{23}$$

where $\alpha = Prob[Rule2]$. Since $\psi_i \in \Delta^{\beta^k_{-i}}(\Theta\times\hat{T}_{-i})$. By Definition 10, there exists $y(\cdot)\in Y_i$ such that

$$\sum_{\theta,\hat{t}_{-i}}\psi_i(\theta,\hat{t}_{-i})u_i(y(\hat{t}_{-i}),\theta) > \sum_{\theta,\hat{t}_{-i}}\psi_i(\theta,\hat{t}_{-i})u_i(\mathfrak{y}_i^{\beta_i}(\hat{t}_{-i}),\theta). \tag{24}$$

Lemma 6 implies that there exists $y_i\in\Delta(A)$ such that

$$\sum_{\theta\in\Theta}\phi_i(\theta)u_i(y_i,\theta) > \sum_{\theta\in\Theta}\phi_i(\theta)u_i(\hat{y}_i,\theta) \tag{25}$$

Since $m_i \in BR_i(marg_{\Theta\times M_{-i}}\pi_i)$, we can conclude that

$$\sum_{\theta,\hat{t}_{-i}}\psi_i(\theta,\hat{t}_{-i})u_i(m_i^3(\hat{t}_{-i}),\theta) \geq \sum_{\theta,\hat{t}_{-i}}\psi_i(\theta,\hat{t}_{-i})u_i(y(\hat{t}_{-i}),\theta) > \sum_{\theta,\hat{t}_{-i}}\psi_i(\theta,\hat{t}_{-i})u_i(\mathfrak{y}_i^{\beta_i}(\hat{t}_{-i}),\theta). \tag{26}$$

$$\sum_{\theta\in\Theta}\phi_i(\theta)u_i(m_i^4,\theta) \geq \sum_{\theta\in\Theta}\phi_i(\theta)u_i(y_i,\theta) > \sum_{\theta\in\Theta}\phi_i(\theta)u_i(\hat{y}_i,\theta). \tag{27}$$

26 and 26 implies that $U_i(m_i,\pi_i)$ is strictly increasing in m_i^2, which is a contradiction. Thus, we have that $m_i^2 = 1$.

Now we show that $\pi_i \in \Delta^{\tilde{\Sigma}_{-i}^{\beta^k_{-i}}}(\Theta\times T_{-i}\times M_{-i})$. Suppose not then either Rule 2 applies with some other agent $j \neq i$ or Rule 3 applies. We focus on the case when only Rule 2 applies. By the definition of g, for every $(\theta, m_{-i}) \in supp(\underset{\Theta\times M_{-i}}{marg}\pi_i)$, it holds that

$$g(\cdot, m_{-i}) = (1 - \frac{1}{m_j^2+1})m_j^3(m_{-j}^1) + \frac{1}{m_j^2+1}\mathfrak{y}_j^{\beta_j}(m_{-j}^1) \tag{28}$$

for some $j \neq i$, where

$$\mathfrak{y}_j^{\beta^{j(i)}}(j_{-i}) = (1-\varepsilon)\,\bar{y}_j^{\beta^{k(j)}}(t_{-j}) + \varepsilon\hat{y} \tag{29}$$

Define $\tilde{g}$ as

$$\tilde{g}(\cdot, m_{-i}) = (1 - \frac{1}{m_j^2+1})m_j^3(m_{-j}^1) + \frac{1}{m_j^2+1}\tilde{\mathfrak{y}}_j^{\beta_j}(m_{-j}^1) \tag{30}$$

where $\tilde{\mathfrak{y}}_j^{\beta^{j(i)}}(t_{-j}) = (1-\varepsilon)\,\bar{y}_j^{\beta^{k(j)}}(t_{-j}) + \varepsilon y_i$. Define $\hat{m}_i^4$ as follows

$$\hat{m}_i^4 = \sum marg_{\Theta\times M_{-i}}\pi_i(\theta, m_{-i})\tilde{g}(\cdot, m_{-i}). \tag{31}$$

Since the agent obtains strictly higher utility under $\tilde{g}(\cdot, m_{-i})$ than $g(\cdot, m_{-i})$ for every $(\theta, m_{-i}) \in supp(\underset{\Theta\times M_{-i}}{marg}\pi_i)$, agent i can announce a message $\hat{m}_i$ with $\hat{m}_i^4$ as defined above and $\hat{m}_i^2 > 1$. In this case Rule 3 is triggered and the agent obtains strictly higher utility. Since the gain is obtained point wise in the $supp(\underset{\Theta\times M_{-i}}{marg}\pi_i)$, we arrive at a contradiction.

□

Lemma 9. For all $k \geq 0$, all $i \in \mathcal{I}\left(\beta^k\right)$ and all $t_i \in T_i$, if $m_i \in S_i^{k+1,\mathcal{M},\mathcal{T}}(t_i)$, then $m_i^2 = 1$ and $m_i^1 \in \beta_i^{k+1}(t_i)$.

Proof. Let us proceed by induction over k. Let $k = 0$. Assume that $i \in \mathcal{I}(\beta^0)$ and fix any $t_i \in T_i$. Assume that $m_i \in S_i^{1,\mathcal{M},\mathcal{T}}(t_i)$. We show that $m_i^2 = 1$ and $m_i^1 \in \beta_i^1(t_i)$. Since $m_i \in S_i^{1,\mathcal{M},\mathcal{T}}(t_i)$, it follows from (1) that there exists $\pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{\Sigma_{-i}^{\beta^0_{-i}}}(\Theta \times T_{-i} \times M_{-i})$, where $\Sigma_{-i}^{\beta^0_{-i}} = S_{-i}^{0,\mathcal{M},\mathcal{T}} = M_{-i}$, such that $m_i \in BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i|\mathcal{M}\right)$. Since $m_i \in BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i|\mathcal{M}\right)$ and $\pi_i \in$

$\Delta^{\Sigma_{-i}^{\beta^0_{-i}}}(\Theta \times T_{-i} \times M_{-i})$, Lemma 8 implies that $m_i^2 = 1$ and

$$\pi_i \in \Delta^{\tilde{\Sigma}_{-i}^{\beta^0_{-i}}}(\Theta \times T_{-i} \times M_{-i}).$$

Thus, we have that

$$\sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta^0_{-i}}(t_{-i})} \pi_i [\theta, t_{-i}, m_{-i}] = 1.$$

Let us define $\nu_i(t_i) \in \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be defined by

$$\nu_i(t_i)\left[\theta, t_{-i}, m_{-i}^1\right] = \sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta^0_{-i}}\left(m_{-i}^1\right)} \pi_i [\theta, t_{-i}, m_{-i}]. \quad (32)$$

By definition, we can see that $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times M_{-i}^1\right) \cap \Delta^{\beta^0_{-i}}\left(\Theta \times T_{-i} \times M_{-i}^1\right)$. Since $m_1^2 = 1$, then *Rule 1* applies with probability 1, and so

$$\begin{aligned} \textstyle\sum_{(\theta, m_{-i}) \in \Theta \times M_{-i}} \left(\operatorname{marg}_{\Theta \times M_{-i}} \pi_i [\theta, m_{-i}]\right) u_i (g(m_i, m_{-i}), \theta) &= \\ \textstyle\sum_{(\theta, m_{-i}) \in \Theta \times M_{-i}} \left(\operatorname{marg}_{\Theta \times M_{-i}} \pi_i [\theta, m_{-i}]\right) u_i \left(f\left(m_i^1, m_{-i}^1\right), \theta\right), & \end{aligned} \quad (33)$$

and so, by (32),

$$\begin{aligned} \textstyle\sum_{(\theta, m_{-i}) \in \Theta \times M_{-i}} \left(\operatorname{marg}_{\Theta \times M_{-i}} \pi_i [\theta, m_{-i}]\right) u_i \left(f\left(m_i^1, m_{-i}^1\right), \theta\right) &= \\ \textstyle\sum_{\left(\theta, m_{-i}^1\right) \in \Theta \times M_{-i}^1} \left(\operatorname{marg}_{\Theta \times M_{-i}^1} \nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i \left(f\left(m_i^1, m_{-i}^1\right), \theta\right). & \end{aligned} \quad (34)$$

Moreover, since $m_i \in BR_i\left(\operatorname{marg}_{\Theta \times M_{-i}} \pi_i | \mathcal{M}\right)$ and since, moreover, player i can never induce *Rule 3*, it follows from the definition of g that

$$\begin{aligned} \textstyle\sum_{\left(\theta, m_{-i}^1\right) \in \Theta \times M_{-i}^1} \left(\operatorname{marg}_{\Theta \times M_{-i}^1} \nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i \left(f\left(m_i^1, m_{-i}^1\right), \theta\right) &\geq \\ \textstyle\sum_{\left(\theta, m_{-i}^1\right) \in \Theta \times M_{-i}^1} \left(\operatorname{marg}_{\Theta \times M_{-i}^1} \nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i \left(m_i^3\left(m_{-i}^1\right), \theta\right), & \end{aligned} \quad (35)$$

for all $m_i^3 \in Y_i^*$. Since Y_i^* is a countable, dense subset of Y_i^f and since u_i is continuous, we have that the inequality in (35) holds for all $m_i^3 \in Y_i^f$. Since $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times M_{-i}^1\right) \cap \Delta^{\beta_{-i}^0}\left(\Theta \times T_{-i} \times M_{-i}^1\right)$ and since, moreover, the inequality in (35) holds for all $m_i^3 \in Y_i^f$ and $m_i^1 \in \beta_i^0(t_i) = T_i$, it follows from (4) that $m_i^1 \in \beta_i^1(t_i)$. Since the choice of (i, t_i) was arbitrary, we have that the stament holds for all $i \in \mathcal{I}\left(\beta^k\right)$ and all $t_i \in T_i$ when $k = 0$.

Suppose that for some $k \geq 0$, and for all $i \in \mathcal{I}\left(\beta^k\right)$ and all $t_i \in T_i$, if $m_i \in S_i^{k+1,\mathcal{M},\mathcal{T}}(t_i)$, then $m_i^2 = 1$ and $m_i^1 \in \beta_i^{k+1}(t_i)$. Fix any $i \in \mathcal{I}\left(\beta^{k+1}\right)$ and any $t_i \in T_i$. Suppose that $m_i \in S_i^{k+2,\mathcal{M},\mathcal{T}}(t_i)$. We show that $m_i^2 = 1$ and $m_i^1 \in \beta_i^{k+2}(t_i)$.

Since $m_i \in S_i^{k+2,\mathcal{M},\mathcal{T}}(t_i)$, it follows from (1) that there exists $\pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{S_{-i}^{k+1,\mathcal{M},\mathcal{T}}}(\Theta \times T_{-i} \times M_{-i})$ such that $m_i \in BR_i\left(\mathrm{marg}_{\Theta \times M_{-i}}\pi_i|\mathcal{M}\right)$. To apply Lemma 8, we need to show that $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^{k+1}}}(\Theta \times T_{-i} \times M_{-i})$. This can be done by showing that

$$S_{-i}^{k+1,\mathcal{M},\mathcal{T}} \subseteq \Sigma_{-i}^{\beta_{-i}^{k+1}}. \tag{36}$$

Fix any $j \in \mathcal{I} \backslash \{i\}$. We proceed according to whether $j \in \mathcal{I}\left(\beta^k\right)$ or not.

Suppose that $j \in \mathcal{I}\left(\beta^k\right)$. Fix any $t_j \in T_j$ and any $m_j \in S_j^{k+1,\mathcal{M},\mathcal{T}}(t_j)$. The inductive hypothesis implies that $m_j^2 = 1$ and $m_j^1 \in \beta_j^{k+1}(t_j)$. It follows from (16) that $m_j \in \Sigma_j^{\beta_j^{k+1}}(t_j)$. Since the choice of $(j, t_j) \in \mathcal{I}\left(\beta^k\right) \times T_j$ was arbitrary, it follows that $S_j^{k+1,\mathcal{M},\mathcal{T}} \subseteq \Sigma_j^{\beta_j^{k+1}}$ for all $j \in \left(\mathcal{I} \cap \mathcal{I}\left(\beta^k\right)\right) \setminus \{i\}$.

Suppose that $j \in \mathcal{I}^c\left(\beta^k\right)$. Since f satisfies IIM on $\mathcal{T}$, Lemma 4 implies that $\beta_j^{k+1} = \beta_j^k = \bar{\beta}_j$. Then, it follows from (16) that $m_j \in \Sigma_j^{\beta_j^{k+1}}(t_j)$. Again, since the chocie of $j \in \mathcal{I}^c\left(\beta^k\right)$ was arbitrary, we conclude that (36) holds.

Since $m_i \in BR_i\left(\mathrm{marg}_{\Theta \times M_{-i}}\pi_i|\mathcal{M}\right)$ and since $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^{k+1}}}(\Theta \times T_{-i} \times M_{-i})$, Lemma 8 implies that $m_i^2 = 1$ and

$$\pi_i \in \Delta^{\tilde{\Sigma}_{-i}^{\beta_{-i}^{k+1}}}(\Theta \times T_{-i} \times M_{-i}).$$

Thus, we have that

$$\sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^{k+1}}(t_{-i})} \pi_i[\theta, t_{-i}, m_{-i}] = 1.$$

Let us define $\nu_i(t_i) \in \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be defined by

$$\nu_i(t_i)\left[\theta, t_{-i}, m_{-i}^1\right] = \sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^{k+1}}\left(m_{-i}^1\right)} \pi_i\left[\theta, t_{-i}, m_{-i}\right]. \tag{37}$$

By definition, we can see that $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times M_{-i}^1\right) \cap \Delta^{\beta_{-i}^{k+1}}\left(\Theta \times T_{-i} \times M_{-i}^1\right)$. Since $m_1^2 = 1$, then *Rule 1* applies with probability 1, and so the equality in (33) holds, and so, by (37),

$$\begin{aligned}\sum_{(\theta,m_{-i}) \in \Theta \times M_{-i}} \left(\mathrm{marg}_{\Theta \times M_{-i}} \pi_i\left[\theta, m_{-i}\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right) &= \\ \sum_{\left(\theta, m_{-i}^1\right) \in \Theta \times M_{-i}^1} \left(\mathrm{marg}_{\Theta \times M_{-i}^1} \nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right).\end{aligned}$$

Moreover, since $m_i \in BR_i\left(\mathrm{marg}_{\Theta \times M_{-i}} \pi_i | \mathcal{M}\right)$ and since, moreover, player i can never induce *Rule 3*, it follows from the definition of g that

$$\begin{aligned}\sum_{\left(\theta, m_{-i}^1\right) \in \Theta \times M_{-i}^1} \left(\mathrm{marg}_{\Theta \times M_{-i}^1} \nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right) &\geq \\ \sum_{\left(\theta, m_{-i}^1\right) \in \Theta \times M_{-i}^1} \left(\mathrm{marg}_{\Theta \times M_{-i}^1} \nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(m_i^3\left(m_{-i}^1\right), \theta\right),\end{aligned} \tag{38}$$

for all $m_i^3 \in Y_i^*$. Since Y_i^* is a countable, dense subset of Y_i^f and since u_i is continuous, we have that the inequality in (38) holds for all $m_i^3 \in Y_i^f$. Since $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times M_{-i}^1\right) \cap \Delta^{\beta_{-i}^{k+1}}\left(\Theta \times T_{-i} \times M_{-i}^1\right)$ and since, moreover, the inequality in (38) holds for all $m_i^3 \in Y_i^f$, and $m_i^1 \in \beta_i^{k+1}(t_i)$, it follows from (4) that $m_i^1 \in \beta_i^{k+1}(t_i)$, as we sought. Since the choice of (i, t_i) was arbitrary, we have that the stament holds for all $i \in \mathcal{I}\left(\beta^{k+1}\right)$ and all $t_i \in T_i$.

By the principle of mathematical induction, we conclude that the statement holds for all $k \geq 0$. □

Let us show that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. Lemma 7 implies that for all $i \in \mathcal{I}$ and $t_i \in T_i$, $S_i^{\mathcal{M},\mathcal{T}}(t_i) \neq \emptyset$. Thus, part (i) of Definition 4 is satisfied. Moreover, Lemma 7 also implies that $\mathcal{M}$ satisfies the EBRP. Recall that by Lemma 5, we are under the assumption that $\mathcal{I}(\beta^*) = \mathcal{I}$. Moreover, recall that Lemma 2 implies that $\left(\beta^k\right)_{k \geq 0}$ is a monotonic decreasing sequence converging to $\beta^* \in \mathcal{B}$. Fix any $t \in T$ and any $m \in S^{\mathcal{M},\mathcal{T}}(t)$. Lemma 9 implies that $m_i^2 = 1$ and $m_i^1 \in \beta_i^*(t_i)$ for all $i \in \mathcal{I}(\beta^*)$. *Rule 1* implies that $g(m) = f(m^1)$. Since f satisfies IIM on $\mathcal{T}$, it follows that β^* is an acceptable deception on $\mathcal{T}$ for f. This implies that $f(m^1) = f(t)$. Since the choice of $(t, m) \in T \times S^{\mathcal{M},\mathcal{T}}(t)$ was arbitrary, we conclude that part (ii) of 4 is satisfied. Thus, f is ICR-implementable on $\mathcal{T}$.

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