Characterization and Implementation of the Shapley Value for Fair Reordering Problems

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What is the question?

In this research, the authors try to set up a mechanism design, for which there is a service provided for agents one at a time. The agents have an initial priority to enjoy the service and with different waiting costs.

The question here is that: Whether the service providing rule "first-come-first-serves *(fcfs)*" is a better system? If there exists alternatives for social planners to reorder the queue to improve overall utilities?

Why should we care about it?

In the literature, the problems are usually solved without initial priorities, which means that all agents have equal rights to be first-served. In reality, however, people pick ticket numbers and an initial queue is predetermined.

The authors provided COVID-19 vaccination as an active example. Just as the case in Taiwan and most of the other countries, when it comes to the order of vaccination, medical staffs are listed as priorities, following the elderly and high-risk professionals. It is obvious that there is reason behind this arrangement rather than *fcfs*, probably for that those people are much riskier than others to be infected.

What is the answer?

The results can be summarized in 3 parts:

- After defining the queueing problem, the authors proposed a game procedure which extend from the literature.
- Their solution, which is the *Shapley value*, satisfies the characteristics of [I] Efficiency [II] Budget Balance *(BB)* [III] Pareto Indifference *(PI)* [IV] Individual Rationality *(IR)* [V] Independence of Local Position Change *(ILPC)* and [VI] Equal Sharing of Local (dis)Imporvement *(ESLI)*. Also, the solution is unique.
- The authors proposed a non-cooperative game (strategic game) to justify the Shapley value, and the social planner does not require any private information.

How did the author get there?

- In cooperation game, the authors set up axioms and game rules, and then check if the solution meet those axioms.
- In non-cooperation game, the authors set up games and figure out SPNE of the game.

Appendix

Settings

- A finite set of agents: $N \in \mathbb{N}$, $N = \{1, 2, \dots, n\}$ with $n \geq 2$.
- The set of all possible queue: $\sum(N)$.
- A queue $\sigma : N \to \{1, 2, \dots n\}$ is an onto function, for all $\sigma \in \sum(N)$. $\sigma^{\prec}$ denotes the initial queue.
- The cash transfer profile $t \in \mathbb{R}^n$.
- The profile of waiting costs $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n_+$. At the beginning every agent i incurs a waiting cost $(\sigma^{\prec} - 1)\theta_i \geq 0$.
- $S \subset N$ is the coalition within N.
- Preference $u_i(\sigma_i, t_i, \theta_i) = -(\sigma - 1)\theta_i + t_i$.
- The net utility gain $U_i(\sigma_i, t_i; \theta_i) = (\sigma_i^{\prec} - \sigma_i)\theta = u_i(\sigma_i, t_i, \theta_i) + (\sigma_i^{\prec} - 1)\theta_i$.
- A queueing problem with an initial priority is defined by a list $q = (N, \theta, \sigma^{\prec}) \in 2^{\mathbb{N}} * \mathbb{R}_+^{|N|} * \mathbb{N}_+^{|N|}$.
- An allocation for $q \in \mathcal{Q}$ is a list $z = (\sigma, t) \in \mathbb{N}^n * \mathbb{R}^n$.
- The set of all allocations $\sum(q)$, for all $q \in \sum(N)$.
- φ is an allocation rule which associates to each problem $q \in (N, \theta; \sigma^{\prec}) \in \mathcal{Q}$ a non-empty subset $\varphi(q)$ of allocations.

Definitions

1. An allocation $z = (\sigma, t)$ is budget balance *(BB)* if for each profile θ, $\sum_{i \in N} t_i = 0$.
2. An allocation $z = (\sigma, t)$ is feasible if for each profile θ, $\sum_{i \in N} t_i \leq \sum_{i \in N} [\sigma^{\prec} - \sigma]\theta_i$, $\forall \sigma \in N$.
3. An allocation $z = (\sigma, t)$ is efficient for $q = (N, \theta; \sigma^{\prec})$ if $\sigma \in E(\theta)$. Where $E(\theta)$ is the collection for θ if the queue $\sigma = \arg\min_{\sigma' \in \sum(N)} \sum_{i \in N} (\sigma'_i - 1)\theta_i$.
4. An allocation $z = (\sigma, t)$ is individually rational *IR* if for each profile θ, $u_i(\sigma_i, t_i, \theta_i) \geq -(\sigma_i^{\prec} - 1)\theta_i$, for all $i \in N$.

Axioms

1. Efficiency *(E)*: Similar as Definition (3).
2. Budget Balance *(BB)*: Similar as Definition (1).
3. Individual Rationality *(IR)*: Similar as Definition (4).
4. Pareto Indifference *(PI)*: A rule satisfies *PI* iff for each $N \in \mathcal{N}$, each $(\theta, \sigma^{\prec}) \in S^N$, each $(\sigma, t) \in \varphi(\theta, \sigma^{\prec})$, and each $(\sigma', t') \in \varepsilon(\theta, \sigma^{\prec})$ such that for each $i \in N$, $U_i(\sigma', t'; \theta, \sigma^{\prec}) = U_i(\sigma, t; \theta, \sigma^{\prec})$, $(\sigma', t') \in \varphi(\theta, \sigma^{\prec})$.

5. Independence of Local Position Change *(ILPC)*: A rule φ satisfies *ILPC* iff for each $N \in \mathcal{N}$, each pair of problems $(\theta, \sigma^{\prec}), (\theta, \sigma^{\prec'}) \in S^N$ in which $\theta = \theta'$ and there are $i, j \in N$, such that $(\sigma_i^{\prec} - \sigma_i^{\prec'})(\sigma_j^{\prec} - \sigma_j^{\prec'}) = -1$ and for each $k \in N \backslash \{i, j\}$, $\sigma_k^{\prec} = \sigma_k^{\prec'}$, each $(\sigma, t) \in \varphi(\theta, \sigma^{\prec})$, and each $(\sigma', t') \in \varphi(\theta', \sigma^{\prec'})$, $U_k(\sigma, t; \theta, \sigma^{\prec}) = U_k(\sigma', t'; \theta', \sigma^{\prec'})$, for all $k \in N \backslash \{i, j\}$.

6. Equal Sharing of Local (dis)Improvement *(ESLI)*: A rule φ satisfies *ESLI* iff for each $N \in \mathcal{N}$, each pair of problems $(\theta, \sigma^{\prec}), (\theta', \sigma^{\prec'}) \in S^N$ in which $\theta = \theta'$ and there are $i, j \in N$ such that $(\sigma_i^{\prec} - \sigma_i^{\prec'})(\sigma_j^{\prec} - \sigma_j^{\prec'}) = -1$ and for each $k \in N \backslash \{i, j\}$, $\sigma_k = \sigma_k'$, each $(\sigma, t) \in \varphi(\theta, \sigma^{\prec})$, and each $(\sigma', t') \in \varphi(\theta', \sigma^{\prec'})$,

$$
\begin{aligned}
& U_i(\sigma, t; \theta, \sigma^{\prec}) - U_i(\sigma', t'; \theta', \sigma^{\prec'}) \\
=& U_j(\sigma, t; \theta, \sigma^{\prec}) - U_j(\sigma', t'; \theta', \sigma^{\prec'}) \\
=& \frac{1}{2}(\sigma_i^{\prec} - \sigma_i^{\prec'})(\theta_i - \theta_j)
\end{aligned}
$$