Incentive Constrained Risk Sharing, Segmentation, and Asset Pricing[*]

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Abstract

Incentive problems make securities' payoffs imperfectly pledgeable. Introducing these problems in an otherwise canonical general equilibrium model yields a rich set of implications. Security markets are endogenously segmented. There is a basis going always in the same direction: the price of any risky security is lower than that of the replicating portfolio of Arrow securities. Equilibrium expected returns are concave in consumption betas, in line with empirical findings. As the dispersion of consumption betas of the risky securities increases, incentive constraints are relaxed and the basis reduced. When hit by adverse shocks, relatively risk tolerant agents sell their safest securities.

Keywords: General Equilibrium, Asset Pricing, Collateral Constraints, Endogenously Incomplete Markets.

1 Introduction

Financial markets allow agents to share risk by trading contingent claims, such as futures, options or Credit Default Swaps. These contingent claims are promises to make payments in the future, if some states occur. To deliver on these promises, agents must have assets generating resources they can use to pay counterparties. We refer to these promise-backing assets as collateral. This is a broad definition of collateral, encompassing securities and real assets.

Collateral, however, is imperfectly pledgeable, because when an agent defaults it is costly and difficult for creditors to seize and realize the value of his assets. Because creditors want to avoid the cost of default, debtors can ex post renegotiate liabilities, down to the collateral value minus bankruptcy costs (as in Kiyotaki and Moore, 1997; Kiyotaki, 1998). Ex ante, this sets a limit to what can be credibly pledged. Imperfect collateral pledgeability limits the extent to which agents can credibly promise contingent payments, i.e., issue contingent claims, to share risk in financial markets.

Costs of default are significant for financial firms even when there are "safe harbor" provisions, as documented by Fleming and Sarkar (2014) and Jackson, Scott, Summe and Taylor (2011) in case studies of the bankruptcy of Lehman Brothers Holdings Inc. For instance, Fleming and Sarkar (2014, p. 193) write that "it has been alleged that Lehman (...) failed to segregate collateral", that creditors to these claims "were unable to make recovery through the close-out netting process and became unsecured creditor to the Lehman estate", and that "counterparties did not know when their collateral would be returned to them, nor did they know how much they would recover given the deliberateness and unpredictability of the bankruptcy process."

The goal of this paper is to study the implications of imperfect pledgeability for risk sharing and asset pricing. To conduct this analysis, we consider a canonical one-period general equilibrium model with a single consumption good. At time 0, competitive risk averse agents are endowed with shares of securities in positive aggregate supply (or "Lucas trees") generating consumption good output at time 1. Lucas trees are risky and heterogenous: time-1 output varies across states of nature and across trees. The agents can take long and short positions in these trees and while doing so buy or sell the whole vector of outputs generated by a tree in all the possible states of nature. Agents, however, can also take long and short positions in a complete

set of Arrow securities in zero net supply. Any long security positions, either in trees or in Arrow securities, can be used as collateral.

If collateral was perfectly pledgeable, the first best would be attained in equilibrium. Agents would use the output from their collateral to make the payments they promised. This would enable agents to achieve efficient insurance. In this complete market consumption-CAPM world, only the risk associated with aggregate output would be priced. Finally, agents would be indifferent between holding a tree and a corresponding replicating portfolio of Arrow securities, since both would have the same price and the same payoff. As a result, the allocation of trees would be indeterminate.

In contrast, we analyze equilibrium with imperfectly pledgeable collateral. Incentive compatibility implies that an agent's pledgeable income in a given state is only a fraction of his long positions' payoff in that state. This limits the state contingent transfers agents can promise one another and hence constrains the provision of insurance. Thus, even though agents can trade a complete set of Arrow securities, and even if the span of tree payoff is complete, risk sharing can be imperfect and the market endogenously incomplete. We show in this context that equilibrium exists and is constrained Pareto optimal (constrained optimality of equilibrium obtains because, in a one-period model, prices do not appear in incentive compatibility constraints.)

Equilibrium risk sharing is imperfect only if the first best cannot be achieved with only long positions in trees. That is, risk sharing is imperfect only if agents must establish short positions to share risk. Without short positions, i.e., without liabilities, imperfect pledgeability is not an issue. Because short positions create incentive problems, an equilibrium allocation, which is constrained Pareto optimal, minimises the issuance of Arrow securities. When short positions arise in equilibrium, they concern Arrow securities, not trees. This is because shorting trees creates unnecessary liabilities (and therefore tighten incentive constraints), which selling Arrow securities can avoid. Thus, in equilibrium, agents hold long positions in trees generating payoffs as close as possible to their desired state-contingent consumption profile. Correspondingly different agents, with different desired consumption profiles, hold different trees. Equivalently, different trees are held by different clienteles, i.e., the market is endogenously segmented. Segmentation arises even though there is a complete set of Arrow securities, because of endogenous market incompleteness due to incentive constraints.

Incentive constraints also affect asset pricing: The price of a tree is equal to the value of its output eval-

uated at Arrow securities prices, minus the shadow incentive cost of the agent holding it. This implies there is a basis between the price of a tree and that of the replicating portfolios of Arrow securities. Furthermore, the basis always goes in the same direction, as each tree is priced *below* its replicating portfolio of Arrow securities. This is striking because we assume trees and Arrow securities are equally pledgeable.

The basis is a deviation from the Law of One Price but does not constitute an arbitrage opportunity. In order to conduct an arbitrage trade, an agent would need to sell Arrow securities and use the proceeds to buy the tree. This is precluded by the incentive constraint: if the agent sold these Arrow securities, this would increase his liabilities, leading to a violation of his incentive constraint.

Another way to grasp the intuition for the basis is to compare the ease with which the payoffs of trees and portfolios of Arrow securities can be stripped: To strip the tree payoffs one needs to issue Arrow securities collateralized by the tree, which is costly in terms of incentives. In contrast, the payoffs of a portfolio of long positions in Arrow securities can be stripped, by selling some of these securities, without any incentive cost. Now the ease to strip payoffs is valued in the market because it helps agents structuring portfolios that fit their desired consumption profiles. Hence a tree is valued less than its replicating portfolio of Arrow securities.

Following the same logic, there is a basis between the price of a tree and that of a replicating portfolio of long positions in trees and Arrow securities. For example our model predicts, in line with empirical evidence, that the price of a convertible bond is lower than that of the replicating portfolio of the straight bond and the call on the issuer's stock. Likewise, in a binomial-tree version of our model, the price of a portfolio made up of a call option and a bond is greater than that of the underlying asset. This is in line with empirical evidence on option pricing.[1]

Moreover, equilibrium expected excess returns reflect two premia. The first premium increases in the covariance between the tree payoff and the consumption of an unconstrained agent, whose identity varies across states. The second premium increases with the covariance between the tree payoff and the shadow price of the incentive compatibility constraint of the agents holding it.

To further illustrate equilibrium properties, we consider the simple case in which there are two states,

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[1]For example this observation corresponds to the empirical finding of Longstaff (1995) who writes that: "because an option can be viewed as a levered position in the underlying asset, the results suggests that it is more expensive to purchase the stock via the option market than in the stock market." See also Rubinstein (1994) and Bates (2000), among many others.

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two agents' types with different relative risk aversion, and an arbitrary distribution of trees. In equilibrium, as in the first best (but to a lower extent), the consumption share of the more risk tolerant agent is smaller in the bad state than in the good state. That is, the more risk tolerant agent offers some insurance to the more risk averse agent. To implement this allocation, the more risk tolerant agent sells (resp. buys) Arrow securities that pay in the bad (resp. good) state. Hence his incentive compatibility constraint binds in the bad state and is slack in the good state. To mitigate incentive problems, the more risk averse agent holds trees with relatively large payoff in the bad state, so that he needs to purchase less Arrow securities from the risk tolerant agent. That is, the more risk averse agent holds trees with low consumption beta, while, by market clearing, the more risk tolerant agent holds trees with high consumption beta. This is a deviation from the two-fund separation principle, which illustrates how segmentation arises in equilibrium in our model. In this simple case a rich set of implications obtains.

First, expected excess returns are concave in consumption betas, in line with Black (1972) and recent evidence by Frazzini and Pedersen (2014) and Hong and Sraer (2016). That is, an intermediate beta tree is valued less than the replicating combination of a high and low beta tree, which illustrates the above stated general principle that a tree is valued less than a replicating portfolios of long positions in trees.

Second, holding aggregate risk and aggregate pledgeable income constant, the distribution of aggregate output across trees in each state matters for equilibrium outcomes. Namely, incentive constraints are less likely to impact equilibrium if the distribution of security betas is more dispersed. This is because the availability of low and high beta trees increases the ease with which agents can construct portfolios of trees with payoffs close to their desired consumption profiles. In practice, the dispersion of security betas might be lower when many firms are conglomerates; when the productive sector relies on a relatively small number of technologies; or when many corporations are financially distressed so that safe assets are scarce.

Third, our analysis sheds light on the consequences of shocks worsening incentive problems. Suppose the more risk tolerant agent is subject to a negative shock, increasing his shadow cost of holding assets. The agent sells his least risky holdings, for which his comparative advantage is the lowest. At the same time, the basis increases for all the assets initially held by the agent, so that there is comovement among these assets.

Literature: Our model is in line with the limited commitment literature, see Kehoe and Levine (1993, 2001), Holmström and Tirole (2001), Alvarez and Jermann (2000), Chien and Lustig (2009) and Gottardi and Kubler (2015). In these papers, and also in ours, while there is a complete set of Arrow securities, incentive constraints prevent full risk-sharing. However, these papers assume that some income (interpreted either as labour or corporate income) cannot be pledged nor traded, but tradeable assets can be pledged. In contrast, we assume that all income is generated by assets that can be traded and imperfectly pledged.[2] This difference in assumption generates a difference in results: In the limited commitment literature, the Law of One Price holds.[3] In contrast, our model exhibits equilibrium deviations from the Law of One Price.

Deviations from the Law of One Price have been obtained by another important strand of literature, in particular Hindy and Huang (1995), Aiyagari and Gertler (1999), Gromb and Vayanos (2002, 2017), Coen-Pirani (2005), Fostel and Geanakoplos (2008), Gârleanu and Pedersen (2011), Geanakoplos and Zame (2014), and Brumm, Grill, Kubler and Schmedders (2015). That literature differs from our paper and from the limited commitment literature in that it specifies plausible but exogenous financial constraints. In contrast, in our paper and in the limited commitment literature, financial constraints stem from the incentive compatibility condition that the agent must prefer to hold his promises rather than deviating.[4] Thus, the equilibrium allocation can be interpreted as the outcome of an optimal contracting process. Endogenizing constraints, moreover, yields new results on segmentation, bases, and equilibrium returns.

Fostel and Geanakoplos (2008), Geanakoplos and Zame (2014), Brumm, Grill, Kubler and Schmedders (2015), Geerolf (2015), Gromb and Vayanos (2002, 2017), and Lenel (2017) also analyze general equilibrium under collateral constraints. In that literature, each financial promise must be backed by its own collateral.[5] In our framework, by contrast, the constraint applies to the portfolio of assets and Arrow securities of an agent, in line with the practice of portfolio margining.[6] As mentioned above, a key finding in our framework

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[2]Rampini and Vishwanathan (2017) also study risk-sharing under financial constraints (between a risk averse hedger and a risk neutral intermediary, via state-contingent debt). Their analysis, which does not consider the trading of assets, differs from ours, which focuses on pricing the cross section of assets.

[3]For example, Alvarez and Jermann (2000) write (on page 776): "The price of an arbitrary asset is calculated by adding up the prices of the corresponding contingent claims." Likewise, in Holmström and Tirole (2001), there is a single vector of state prices that is used to calculate the value of all claims, either corporate or non-corporate (see equation (20) and (23) on page 1849).

[4]In this context payoffs in case of deviation are explicitly specified. For example in Alvarez and Jermann (2000) agents must revert to autarky, while in Chien and Lustig (2009) the agents' holdings of a Lucas tree are seized, and in our model a fraction of the output of all long security positions held by the agent is seized.

[5]So the same asset, generating strictly positive output in two states, cannot be used to collateralize the issuance of two Arrow securities, promising payments in these two states.

[6]For example, on http://www.cboe.com/products/portfolio-margining-rules, one can read: "The portfolio margining

is that assets trade at a discount relative to replicating portfolios of derivatives. This could seem to contradict the "collateral premium" obtained by these papers. Similarly, this could seem to contradict the "liquidity premium" derived by the new monetarist literature for assets that can be used as means of payment (see, for example Lagos, 2010; Li, Rocheteau and Weill, 2012; Lester, Postlewaite and Wright, 2012; Venkateswaran and Wright, 2013; Jacquet, 2015). There is no contradiction, however: it is simply that the benchmark valuation is not the same for the premium and the basis results. The *premium*, which is the difference between the price of the asset and its counterfactual value based on the marginal utility of the agent holding it, obtains in these papers as well as in ours. Analyzing the *discount* at which real assets trade relative to replicating portfolios of Arrow securities is a contribution of our paper relative to that literature.

Krishnamurthy (2003) also studies general equilibrium under collateral constraints. Our paper is related with Krishnamurthy (2003) because (in both) collateral constraints limit hedging. However, our focus on asset pricing issues, such as risk premia, segmentation and deviation from the Law of One Price, differs from his focus on amplification mechanisms in production economies.

Finally, another related strand of the literature has proposed models of clientele, in which assets are held and priced by the endogenous segment of agents who value them most. This includes, for example, Amihud and Mendelson (1986), Dybvig and Ross (1986) and Sharpe (1991). We make two contributions relative to this earlier work. First, clienteles arise from on a new, incentive-based, economic mechanism. Second, we generate clientele without imposing short-selling constraints on individual assets.

The next section presents our model. Section 3 and 4 present general results on equilibrium. Section 5 presents more specific results, obtained when there are only two states and two types of agents. Proofs are in the online appendix.

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rules have the effect of aligning the amount of margin money ... to the risk of the portfolio as a whole, calculated through simulating market moves up and down, and accounting for offsets between and among all products held..."

2 Model

2.1 Agents

There are two dates $t = 0, 1$. The state of the world ω realizes at $t = 1$ and is drawn from some finite set Ω according to the probability distribution $\{\pi(\omega)\}_{\omega \in \Omega}$, where $\pi(\omega) > 0$ for all ω. The economy is populated by finitely many types of agents, indexed by $i \in I$. The measure of type $i \in I$ agents is normalized to one. Agents of type $i \in I$ have Von Neumann Mortgenstern utility

$$U_i(c_i) \equiv \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i(\omega) \right]$$

over time $t = 1$ state-contingent consumption, $c_i \equiv \{c_i(\omega)\}_{\omega \in \Omega}$. We take the utility function to be either linear, $u_i(c) = c$, or strictly increasing, strictly concave, and twice-continuously differentiable over $c \in (0, \infty)$. Without loss of generality, we apply an affine transformation to the utility function $u_i(c)$ so that it satisfies either $u_i(0) = 0$; or $u_i(0) = -\infty$ and $u_i(\infty) = +\infty$; or $u_i(0) = -\infty$ and $u_i(\infty) = 0$. In addition, if $u_i(0) = -\infty$, we assume that the agent's relative risk aversion remains bounded near zero: specifically, that there exists some $\gamma_i > 1$ such that, for all c small enough, $\frac{u_i'(c)c}{|u_i(c)|} \leq (\gamma_i - 1)$.[7]

2.2 Securities

There are two types of securities: trees in positive aggregate supply, and a complete set of Arrow securities in zero net supply.

Trees provide all real resources of the economy. The set of tree types is taken to be a compact interval that we normalize to be $[0, 1]$, endowed with its Borel σ-algebra. The distribution of tree supplies is a positive and finite measure $\bar{N}$ over the set $[0, 1]$ of tree types. We place no restriction on $\bar{N}$: it can be discrete, continuous, or a mixture of both. The payoff of tree $j \in [0, 1]$ in state $\omega \in \Omega$ is denoted by $d_j(\omega) \geq 0$, with at least one strict inequality in some state $\omega \in \Omega$. A technical condition for our existence proof is that, for all $\omega \in \Omega$,

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[7]This implies the Constant Relative Risk Aversion (CRRA) bound $0 \geq u_i(c) \geq Kc^{1-\gamma_i}$ for all c small enough and some negative constant K. We use this technical condition in our existence proof to show that the maximum correspondence of the social planner's problem has a weakly closed graph and that, at points where some of the welfare weights are equal to zero, the maximized social welfare function is continuous in welfare weights. See the proof of Proposition C.1 in the Supplementary Appendix.

$j \mapsto d_j(\omega)$ is continuous. Economically, continuity means that trees are finely differentiated: nearby trees in $[0, 1]$ have nearby characteristics. Continuity is a mild assumption since we do not impose any restriction on the distribution $\bar{N}$ of supplies.[8] As will become clear later, we consider a continuum of trees for two reasons. First, it will demonstrate clearly that our results do not arise because the set of tree payoffs is in some way incomplete. Second, in Section 5, it will make it easier to explicitly characterize patterns of segmentation.

Agent $i \in I$ is initially endowed a strictly positive share, $\bar{n}_i > 0$, in the market portfolio of trees, $\bar{N}$. Agents' shares in the market portfolio add up to one, $\sum_{i \in I} \bar{n}_i = 1$. Agents have no initial endowment of Arrow securities.

At time zero, an agent can take long and short positions in all assets, trees and Arrow securities. The portfolio of agent's i tree positions is denoted by $N_i \equiv (N_i^+, N_i^-)$, where N_i^+ is the portfolio of long positions, and N_i^- is the portfolio of short positions. Both N_i^+ and N_i^- belong to the set $\mathcal{M}_+$ of positive and finite measures over the set of tree types, $[0, 1]$. Likewise, the portfolio of Arrow security positions, long and short, is denoted by $a_i \equiv \{a_i^+(\omega), a_i^-(\omega)\}_{\omega \in \Omega} \in \mathbb{R}_+^{2|\Omega|}$.

2.3 Incentive Constraints

At time one, an agent can strategically default on the contractual obligations created by short positions in trees and Arrow securities. As discussed in the introduction in the Lehman Brother's case, when counterparties default, collateral recovery is costly and imperfect. Debtors can take advantage of such imperfect recoverability to renegotiate liabilities. This opportunity to renegotiate limits the repayments agents can credibly promise to make.

To capture this process in the simplest possible way, we assume an agent can make a take-it-or-leave-it offer to his creditors, who, if they refuse, can only recover a fraction $1 - \delta \in (0, 1]$ of the agent's long positions in trees and Arrow securities. So the agent can always renegotiate his liabilities down to a fraction $1 - \delta$ of his total long positions. Therefore, this is the maximum amount he can credibly promise to repay, i.e., his

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[8]For example, our model nests a standard specification with finitely many trees $k \in \{1, \ldots, K\}$, with state-contingent payoff $D^{(k)}(\omega)$ and positive supplies $S^{(k)} > 0$. Indeed, this discrete specification obtains by fixing a finite sequence $j_1 < j_2, \ldots < j_K$, choosing any continuous function $j \mapsto d_j(\omega)$ such that $d_{j_k}(\omega) = D^{(k)}(\omega)$ for $k \in \{1, \ldots, K\}$, and letting $\bar{N}$ be a discrete distribution with atoms at $j_1 < j_2, \ldots < j_K$, $\bar{N}_{j_k} - \bar{N}_{j_k-} = S^{(k)}$.

pledgeable income.[9] Correspondingly, we impose the following incentive compatibility constraint

$$\int d_j(\omega) \, dN_{ij}^- + a_i^-(\omega) \le (1-\delta) \left[\int d_j(\omega) \, dN_{ij}^+ + a_i^+(\omega) \right],$$ (1)

for each $(i, \omega) \in I \times \Omega$, and where integrals are taken over the set of tree types, $j \in [0,1]$.[10] As shown by equation (1), the state-contingent payoff of all long positions serves as collateral for the state-contingent liabilities created by short positions. But the maximum liability the agent can credibly promise to repay, i.e., the pledgeable income, is lower than the face value of the collateral. The wedge between the two can be interpreted as a haircut and is increasing in δ. Haircuts are not imposed on an individual security basis, but at the level of the aggregate position, or portfolio of the agent. This is in line with the practice of "portfolio margining."[11]

2.4 Definition of Equilibrium

A *price system* for trees and Arrow securities is a pair (p, q), where $p : j \mapsto p_j$ is a continuous and strictly positive function for the price of tree j and $q = \{q(\omega)\}_{\omega \in \Omega}$ is a strictly positive vector in $\mathbb{R}^{|\Omega|}$ for the prices of Arrow securities.[12]

Given the price system (p, q), the time-zero budget constraint for agent i is:

$$\int p_j \left[dN_{ij}^+ - dN_{ij}^- \right] + \sum_{\omega \in \Omega} q(\omega) \left[a_i^+(\omega) - a_i^-(\omega) \right] \le \bar{n}_i \int p_j \, d\bar{N}_j,$$ (2)

where the integrals are taken over the set of tree types, $j \in [0,1]$. At time one, agent i's consumption must

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[9]Since there is a full set of Arrow securities, promising more than this maximum and subsequently defaulting would not complete the market and expand the agent's consumption possibilities. So, unlike in Geanakoplos and Zame (2014), there is no default on the equilibrium path.

[10]Equation (1) is a renegotiation-proofness condition: all agents anticipate that an allocation that would not satisfy (1) would be renegotiated to an allocation satisfying (1). See Appendix A for an explicit derivation.

[11]In the main body of the paper we assume for simplicity that δ is constant across agents and securities. As shown in Appendix C, our proof of existence and our asset pricing results generalize to the case in which δ depends on the identity i of the agent and of the type j of the security.

[12]Restricting attention to strictly positive prices is clearly without loss of generality: indeed, non-positive prices cannot be the basis of an equilibrium because they would result in infinite demand. We also assume that the price functional admits a natural dot-product representation based on a continuous function, p_j, of tree type. This entails some loss of generality, as the price functional should in principle be some arbitrary continuous linear functional, which may not have a dot-product representation. However, we show in Proposition 3, that there always exists an equilibrium in which the price functional admits this representation.

satisfy:

$$c_i(\omega) = \int d_j(\omega) \left[dN_{ij}^+ - dN_{ij}^-\right] + a_i^+(\omega) - a_i^-(\omega). \tag{3}$$

The problem of agent i is to maximize $U_i(c_i)$ with respect to a plan for consumption, $c_i = \{c_i(\omega)\}_{\omega \in \Omega}$, long and short portfolios of trees, $N_i = (N_i^+, N_i^-)$, and of Arrow securities, $a_i = \{a_i^+(\omega), a_i^-(\omega)\}_{\omega \in \Omega}$, subject to the incentive constraints (1) for all $\omega \in \Omega$, and to the time-zero and time-one budget constraints (2) and (3). An allocation is a collection of plans $(c_i, N_i, a_i)_{i \in I}$ for all agents. A *security market equilibrium* is, then, an allocation, $(c_i, N_i, a_i)_{i \in I}$, and a price system (p, q), such that for all $i \in I$, (c_i, N_i, a_i) solves agent's i problem given prices and all asset markets clear, that is:

$$\sum_{i \in I} N_i^+ = \bar{N}_i + \sum_{i \in I} N_i^- \tag{4}$$

$$\sum_{i \in I} a_i^+(\omega) = \sum_{i \in I} a_i^-(\omega) \text{ for all } \omega \in \Omega. \tag{5}$$

Our formulation of the agent's problem clearly shows that, since there is a complete set of Arrow securities, an agent can always avoid any incentive problems by selling all of his tree endowment, and only purchase a portfolio of Arrow securities corresponding to his desired state-contingent consumption profile. But this, of course, cannot be part of an equilibrium: if all agents sold all of their trees, the market clearing condition for trees, (4), would not hold. This simple observation intuitively implies our result about bases: for the tree market to clear, the price of trees will have to fall below that of Arrow securities. This also highlights that, in our model, binding incentive compatibility constraints is ultimately a general equilibrium phenomenon. This is in contrast with earlier models in which non-pledgeable income is not tradeable: in these environments, binding incentive constraints would already arise in partial equilibrium contract-theoretic settings.

3 Ruling Out Short Positions in Trees

In this section, we show that, in a security market equilibrium, two types of positions are weakly suboptimal: short tree positions, $N_{ij}^-([0, 1]) > 0$, and simultaneous non netted long and short positions in the same Arrow security, $a_i^+(\omega) > 0$ and $a_i^-(\omega) > 0$. Because these positions are weakly suboptimal, equilibrium consumption

and prices are not affected by whether they are allowed or not. In that sense they are irrelevant. This result allows us to define an equilibrium concept in the spirit of the standard Arrow-Debreu equilibrium (see Mas-Colell, Whinston and Green, 1995, page 691), ruling out these irrelevant asset positions, and thus imposing simpler budget and incentive constraints, but leading to identical equilibrium consumptions and prices.

Irrelevance of short tree positions, and of non netted positions. Our first result is that any equilibrium consumptions and prices can be supported without short positions in trees and non netted positions.

Lemma 1 *Consider any security market equilibrium* $(c_i, N_i, a_i)_{i \in I}$ *and* (p, q). *Then, there exists another security market equilibrium* $(\hat{c}_i, \hat{N}_i, \hat{a}_i,)_{i \in I}$ *and* $(\hat{p}, \hat{q})$ *such that:* $\hat{c}_i = c_i$, $\hat{N}_i^- = 0$, $\hat{a}_i^+(\omega)\hat{a}_i^-(\omega) = 0$ *for all* $\omega \in \Omega$, $\hat{p} = p$, *and* $\hat{q} = q$.

In the standard perfect and complete market model, Lemma 1 is immediate because agents' constraints and payoffs only depend on net security positions, and because the Law of One Price holds for all securities. Neither condition holds in the presence of incentive constraints, which makes the result non obvious.

To prove Lemma 1, we first construct the candidate equilibrium allocation $(\hat{c}_i, \hat{N}_i, \hat{a}_i)_{i \in I}$ by keeping consumption the same, $\hat{c}_i = c_i$, and by substituting and netting asset positions in (N_i, a_i). Namely, we substitute all short tree positions with short positions in replicating portfolios of Arrow securities. To restore market clearing for trees whose short positions have been eliminated, we scale down agents long positions in those trees and substitute them with long positions in replicating portfolios of Arrow securities. Finally, we net all long and short Arrow positions. Next, we show that, given prices (p, q), the candidate plan $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ remains optimal for each agent i. Since $\hat{c}_i = c_i$, all we need to show is that the candidate plan satisfies the incentive and budget constraints.

The incentive constraints hold for two reasons. First, substitution of tree positions by corresponding positions in replicating portfolio of Arrow securities does not impact incentive constraints, since the latter only depend on the total value of assets or liabilities, and not on their compositions. Second, netting positions in Arrow securities relaxes incentive constraints, since it reduces the total value of liabilities, on the left-hand side of (1), by more than the total pledgeable value of assets, on the right-hand side.

To see why the budget constraint holds, note that although the Law of One Price may not apply to all assets (as will become clear later), it must apply to all trees that, in our construction, are substituted by replicating portfolio of Arrow securities. Indeed, trees are only substituted if they are held short by some agents and long by others. But short tree positions can only be optimal for trees priced weakly above their replicating portfolio, and vice versa for long positions. Taken together, this implies that the Law of One Price must hold for all trees that are held both long and short, and therefore for all trees that, in our construction, are substituted by replicating portfolios.

An equivalent Arrow-Debreu equilibrium concept. Lemma 1 shows that we can assume without loss of generality that agents do not short trees and do not take simultaneous long and short positions in Arrow securities, $a_i^+(\omega)a_i^-(\omega) = 0$. This implies that, in the incentive compatibility constraint, agents never use long positions in Arrow securities as collateral for a short position in the same Arrow security. Hence, for incentive compatibility, it is necessary and sufficient that agents have enough tree collateral to secure their *net* Arrow security positions: $-\left[a_i^+(\omega) - a_i^-(\omega)\right] \leq (1-\delta)\int d_j(\omega)\,dN_{ij}^+$. Expressing $a_i^+(\omega) - a_i^-(\omega)$ in terms of consumption and tree payoff, we obtain the equivalent incentive constraint:

$$c_i(\omega) \geq \delta \int d_j(\omega)\,dN_{ij}^+ \text{ for all } \omega \in \Omega. \tag{6}$$

Next, we make the standard observation that the two sequential budget constraints, (2) and (3), can be consolidated in one single inter-temporal budget constraint:

$$\sum_{\omega \in \Omega} q(\omega)c_i(\omega) + \int p_j\,dN_{ij} \leq \bar{n}_i \int p_j\,d\bar{N}_j + \sum_{\omega \in \Omega} q(\omega)\int d_j(\omega)\,dN_{ij}^+. \tag{7}$$

These observations allow us to define an equivalent concept of *Arrow-Debreu equilibrium*, in which an agent purchases directly state-contingent consumption at time zero, without needing to consider explicitly positions in Arrow securities. Namely, *the problem of agent i is to maximize $U_i(c_i)$* with respect to a plan for consumption and long tree positions, $(c_i, N_i^+) \in \mathbb{R}_+^{|\Omega|} \times \mathcal{M}_+$, subject to the intertemporal budget constraint (7) and the incentive constraints (6). An *allocation* is a collection $(c_i, N_i^+)_{i \in I}$ of consumption plans and long

tree positions for every agent $i \in I$. An *Arrow-Debreu equilibrium* is an allocation $(c_i, N_i^+)_{i \in I}$ and a price system (p, q) such that, for all $i \in I$, (c_i, N_i^+) solves agent's i problem given prices and markets clear:

$$\sum_{i \in I} N_i^+ = \bar{N} \tag{8}$$

$$\sum_{i \in I} c_i(\omega) \leq \sum_{i \in I} \int d_j(\omega) \, dN_{ij}^+ \text{ for all } \omega \in \Omega. \tag{9}$$

We can now state that the concept of Arrow-Debreu equilibrium is equivalent to that of security market equilibrium in the following sense:

Proposition 1 *If (c_i, N_i, a_i) and (p, q) is a security market equilibrium, then there exists an Arrow-Debreu equilibrium $(\hat{c}_i, \hat{N}_i^+)$ and $(\hat{p}, \hat{q})$ such that $\hat{c}_i = c_i$, $\hat{p} = p$ and $\hat{q} = q$. Conversely, if $(\hat{c}_i, \hat{N}_i^+)$ and $(\hat{p}, \hat{q})$ is an Arrow-Debreu equilibrium, then there exists a security market equilibrium, (c_i, N_i, a_i) and (p, q), such that $c_i = \hat{c}_i$, $N_i^+ = \hat{N}_i^+$, $N_i^- = 0$, $p = \hat{p}$, $\bar{N}$-almost everywhere, and $q = \hat{q}$.*

That for each security market equilibrium, there exists an Arrow-Debreu equilibrium with the same consumption and prices follows from Lemma 1. The key reason why the converse holds, is that no agent strictly prefers to move away from the Arrow-Debreu allocation by taking short tree positions or non netted positions in Arrow securities, since, as in Lemma 1, such positions are dominated.[13]

4 Equilibrium analysis

Based on the equivalence result of Proposition 1, we focus, for the remainder of the paper, on Arrow-Debreu equilibria.

4.1 Incentive-Constrained Pareto Optimality and Existence

An allocation $(c_i, N_i^+)_{i \in I}$ is said to be *incentive-feasible* if it satisfies the incentive compatibility constraints (6) for all $(i, \omega) \in I \times \Omega$, and the feasibility constraints (8) and (9). An incentive-feasible allocation $(\hat{c}_i, \hat{N}_i^+)_{i \in I}$

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[13]In the statement of the converse, the equality among prices is said to hold almost everywhere because it holds only for trees in strictly positive supply. Indeed, for securities in zero net supply, the Arrow-Debreu equilibrium does not rule out very high prices (since short sales are not allowed), while in the security market equilibrium such prices cannot arise (otherwise all investors would want to short sell the trees).

Pareto dominates the incentive-feasible allocation (c, N) if $U_i(\hat{c}_i) \geq U_i(c_i)$ for all $i \in I$, with at least one strict inequality for some $i \in I$. An allocation is *incentive-constrained Pareto optimal* if it is incentive-feasible and not Pareto dominated by any other incentive-feasible allocation. In our model, we have:

Proposition 2 *Any Arrow-Debreu equilibrium allocation is incentive-constrained Pareto optimal.*

The reason why optimality obtains in spite of incentive constraints is because prices do not show up in the incentive compatibility condition, so that there are no "contractual externalities".[14] See Prescott and Townsend (1984) and Kehoe and Levine (1993) for other examples of economies in which equilibrium is constrained optimal, and Gottardi and Kubler (2015) for examples in which it is not. Because there are no contractual externalities, the proof of Proposition 2 is similar to its perfect market counterpart: if an equilibrium allocation were Pareto dominated by another incentive feasible allocation, the latter must lie outside the agents' budget set. Adding up across agents leads to a contradiction.

As is standard, the Pareto efficiency result of Proposition 2 is especially useful because it facilitates the proof of equilibrium existence. Namely, following standard steps, we consider the problem of a Planner who assigns Pareto weights $\alpha_i \geq 0$ to each agent $i \in I$, with $\sum_{i \in I} \alpha_i = 1$, and chooses incentive feasible allocations to maximize the social welfare function, $\sum_{i \in I} \alpha_i U_i(c_i)$. The proof of existence then boils down to the problem of finding Pareto weights such that, given agents' initial endowment, the Planner's solution can be implemented in a competitive equilibrium without making any wealth transfers between agents. This leads to:

Proposition 3 *There exists an Arrow-Debreu equilibrium, $(c_i, N_i^+)_{i \in I}$ and (p, q).*

The proof follows arguments found in Negishi (1960), Magill (1981), and Mas-Colell and Zame (1991) with some differences. First, our planner is now subject to incentive compatibility constraints. Second, the commodity space includes portfolios of trees and is therefore infinite dimensional, which makes it harder to apply separation theorems. Proposition 3 addresses these difficulties by explicitly deriving first-order necessary and sufficient conditions for the Planner's problem, and using the associated Lagrange multipliers to construct equilibrium prices.

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[14]While Proposition 2 states that equilibrium is incentive constrained Pareto optimal, we show in Appendix C.3 that equilibrium is unconstrained Pareto optimal when it can be implemented without short positions. Without short positions, incentive compatibility conditions are slack.

While we cannot prove uniqueness in general, we show in Appendix C.4 that uniqueness obtains in a particular case of interest: when there are two types of agents with CRRA utilities, with relative risk aversion less than one.

4.2 First-order conditions

The key implications of our model are obtained based on agents' first-order conditions. Since agents have concave objectives and are subject to finitely-many affine constraints, we can apply the Lagrange multiplier Theorems shown in Section 8.3 and 8.4 of Luenberger (1969) (see Proposition C.4 in the appendix for details). Let λ_i denote the Lagrange multiplier of the intertemporal budget constraint (7) and $\mu_i(\omega)$ the Lagrange multiplier of the incentive compatibility constraint (6). The first-order condition with respect to $c_i(\omega)$ is[15]

$$\pi(\omega)u_i'[c_i(\omega)] + \mu_i(\omega) = \lambda_i q(\omega). \tag{10}$$

The first term on the left-hand side of (10) reflects that an increase in consumption increases utility and the second term reflects that it relaxes the incentive compatibility constraint (6), while the term on the right-hand side reflects that this increase in consumption tightens the budget constraint.

The first-order condition with respect to N_i^+ is

$$\sum_{\omega \in \Omega} \mu_i(\omega)\delta d_j(\omega) \geq \lambda_i \left[\sum_{\omega \in \Omega} q(\omega)d_j(\omega) - p_j \right], \tag{11}$$

with an equality N_i^+–almost everywhere, that is, for almost all trees held by agent i. The left-hand side of (11) reflects that an increase in i's holdings of tree j increases the amount of non-pledgeable dividend $\delta\, d_j(\omega)$, which tightens the incentive constraint (6) for each ω. The right-hand side reflects that an increase in i's holdings of tree j relaxes the intertemporal budget constraint by $\sum_\omega q(\omega)d_j(\omega) - p_j$.

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[15]In principle this condition only holds with an inequality if $c_i(\omega) = 0$, for example when utility is linear. However, we show in the appendix (Proposition C.4) that one can always choose multipliers so that this condition holds at equality.

4.3 Segmentation

Equation (11) can be rewritten

$$p_j \geq \sum_{\omega \in \Omega} q(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \frac{\mu_i(\omega)}{\lambda_i} \delta d_j(\omega) \equiv v_{ij}, \tag{12}$$

with an equality for almost all trees held by agent i, and where v_{ij} can be interpreted as the private valuation of agent i for tree j. The first term of v_{ij}, which is common to all agents, is the present value of the dividends evaluated at state prices. The second term of v_{ij} is agent specific. It reflects the shadow cost incurred by agent i when holding one marginal unit of tree j, thus tightening his incentive constraint. It is the product of the shadow cost of the incentive constraint, $\mu_i(\omega)/\lambda_i$, by the non-pledgeable part of the dividend, $\delta d_j(\omega)$, summed across states.

Equation (12) implies that tree j is held by the agents who value it most, because they have the lowest shadow incentive cost of holding this tree. For these agents, $v_{ij} = p_j$, while for the other agents, who do not hold the tree, $v_{ij} \leq p_j$. Thus, differences in private valuations, driven by differences in shadow incentive costs, can lead to market segmentation. This is the case, in particular, when trees have very different payoffs. For example, suppose there is a tree paying off one unit in state ω and zero otherwise and consider some agent i, whose incentive constraint binds in state ω. Then, there exists another agent i' whose incentive constraint is slack in ω, because the incentive constraint cannot bind for all agents in a given state.[16] Therefore $v_{ij} < v_{i'j}$ and agent i does not hold the tree. By continuity, as shown formally in the next proposition, agent i does not hold any tree with large enough dividend in state ω relative to other states.

Proposition 4 *Suppose the equilibrium is not first best, i.e., $\mu_i(\omega) > 0$ for some $i \in I$ and $\omega \in \Omega$. Then, for any agent of type i, there exists $\varepsilon > 0$ and some state ω such that agent i's incentive constraint binds in state ω and agent i strictly prefers not to hold trees j such that $d_j(\omega')/d_j(\omega) < \varepsilon$ for all $\omega' \in \Omega \setminus \omega$.*

The proposition shows that, under natural conditions, segmentation arises in equilibrium. Moreover, it establishes the general principle that an agent will not hold trees making relatively large payments when her incentive constraint binds.

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[16] Indeed, adding up the incentive constraint (6) across all agents and using market clearing for consumption, one immediately sees that the aggregate incentive constraint must be slack in each state.

The proposition reveals that the equilibrium distribution of trees across agents ultimately depends on the patterns of binding incentive constraints across states and agents. These patterns are difficult to characterize in full generality because they depend on the distribution of risk aversions across agents, and on the distributions of supplies across trees. But this can be done in the context of particular examples, such as the one developed in Section 5 below.

4.4 Asset Pricing

The pricing of risk and incentives. The first order condition with respect to consumption, (10), shows that if the incentive compatibility conditions were slack, the marginal rate of substitution between consumptions in different states would be equal across all agents, as in the standard, perfect and complete markets, model. When incentive compatibility conditions bind, in contrast, marginal rates of substitution differ across agents, reflecting the multipliers of the incentive constraints. This reflects imperfect risk-sharing in markets that are endogenously incomplete due to incentive constraints, as in Kehoe and Levine (2001). Thus the Arrow securities pricing kernel

$$M(\omega) \equiv \frac{q(\omega)}{\pi(\omega)},$$

which in our model prices the Arrow securities, differs from its perfect and complete markets counterpart because in general, there is no agent whose marginal utility is equal to $M(\omega)$ in all states. Instead, as in Alvarez and Jermann (2000), $M(\omega)$ corresponds to the marginal utility of an unconstrained agent, whose type varies from state to state.

Denote

$$A_i(\omega) \equiv \frac{\mu_i(\omega)}{\lambda_i \pi(\omega)},$$

which can be interpreted as the shadow cost of the incentive compatibility constraint of agent i in state ω. Since the first-order condition (12) holds with an equality for almost all trees held by agent i, we obtain:

$$p_j = \max_{i \in I} v_{ij} = \max_{i \in I} \mathbb{E}\left[M(\omega)d_j(\omega)\right] - \delta\,\mathbb{E}\left[A_i(\omega)d_j(\omega)\right]. \tag{13}$$

Equation (13) shows that the price of a tree is the maximum valuation for the asset across all agents. The

price is equal to the present value of the dividends evaluated at Arrow securities prices, minus the shadow incentive-cost of the asset holder.

According to the pricing formula (13), trees with more extreme payoffs, i.e., payoffs close to that of Arrow securities, are less impacted by the shadow incentive-cost of their asset holders and so have relatively higher prices. To see this point, consider a tree in strictly positive supply paying just one unit in state ω and zero otherwise. As argued in Footnote 16, there must exist an agent who is not constrained in state ω. Clearly, equation (12) shows that this agent has the highest valuation for the tree and, correspondingly, equation (13) shows that there is no shadow incentive-cost impounded in the price of the tree. The result that securities with extreme payoffs tend to have higher prices resembles the well-known empirical observation that out-of-the-money calls and puts are expensive (Rubinstein, 1994; Bates, 2000).

Deviations from the Law of One Price. Equation (13) reveals that, if the incentive compatibility constraint of the security holder binds in at least one state, and if the dividend is strictly positive in that state, then the price of the tree is *strictly* smaller than that of the corresponding portfolio of Arrow securities, i.e., there is a basis. One could argue that this constitutes an arbitrage opportunity. However, agents cannot trade on it without tightening their incentive constraint. Thus, the basis between $\mathbb{E}\left[M(\omega)d_j(\omega)\right]$ and the price, p_j, reflects limits to arbitrage induced by incentive constraints.[17]

The basis between trees and replicating portfolios of Arrow securities is a special case of a more general result. Because the maximum operator is convex and v_{ij} is linear in dividends, equation (13) implies that a tree must be priced below *any* replicating portfolio of long positions in trees and/or Arrow securities. The inequality is strict if there is no agent willing to hold all the assets in the replicating portfolio. This is stated formally in the next proposition.

Proposition 5 *For almost all trees j according to $\bar{N}$, consider a replicating portfolio composed of long positions in trees $X \in \mathcal{M}_+$ and Arrow securities $Y \in \mathbb{R}_+^{|\Omega|}$, that is, $d_j(\omega) = \int d_k(\omega)dX_k + Y(\omega)$ for all $\omega \in \Omega$. Suppose there exists no agent willing to hold all the securities in the replicating portfolio, that is,*

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[17]Price differences between trees and Arrow securities do not reflect differences in net supply. They arise because trees pay off in different states, while Arrow securities pay only in one state. To see this, consider a tree in strictly positive supply paying one unit in state ω and zero otherwise. There cannot be a basis between the price of that tree and that of the corresponding Arrow security. If there was, then the agent who is not constrained in state ω (there must be one according to Footnote 16) would sell the Arrow security and buy the tree.

there is no $i \in I$ such that $\int v_{ik} \, dX_k = \int p_k \, dX_k$ and $\mu_i(\omega)Y(\omega) = 0$. Then, tree j is priced strictly below its replicating portfolio:

$$p_j < \int p_k \, dX_k + \sum_{\omega \in \Omega} q(\omega)Y(\omega). \tag{14}$$

The economic intuition is that a tree is a bundle of risks that cannot be traded separately from one another, whereas the portfolio of securities with the same payoff as the tree is a bundle of risks that can be traded separately. An agent prefers to be able to choose among the risks in the bundle, retaining only those he wants to bear, and leaving the remaining risks to other agents. This allocation of risks across agents can be interpreted in terms of clientele.

For example, a convertible bond is a combination of a straight bond and a call option on the issuer's stock. In the language of our model, a convertible bond is a tree with the same payoff as a combination of another tree (the straight bond) with a portfolio of Arrow securities (the call option). Our model implies that, if there are no agents who hold simultaneously the straight bond and the call, then the convertible bond should be priced strictly below the price of the straight bond plus the price of the call. In line with our theory, convertible bonds are in fact priced below the replicating portfolio. This deviation from the Law of One Price is at the root of a popular hedge fund strategy ("convertible arbitrage"), which consists in stripping the convertible bond (Mitchell and Pulvino, 2012). Hedge funds buy the convertible bond, issue the set of securities that replicate the convertible bond, and sell the different securities to different clienteles: debt securities are distributed through prime brokers to money market funds and other buyers of safe securities, while equity risk is distributed to equity investors. The convertible arbitrage strategy is constrained, both in practice and in our theory, because arbitrageurs have a limited ability to issue the securities replicating the convertible bond. As a result, convertible bond cheapness increases when arbitrageurs have greater difficulties issuing liabilities, such as during the 1998 LTCM crisis, the 2005 convertible arbitrage meltdown, and the 2008 credit crisis.

A specific implication of our model is that the basis always goes in the same direction. The price of a tree can be lower than that of the replicating portfolio of trees and/or Arrow securities, but it cannot be higher. If it was higher, an agent holding the tree could sell it and buy the replicating portfolio. That arbitrage trade would be feasible because i) market clearing implies there is at least one agent holding the tree, and

ii) replacing a tree by its replicating portfolio does not tighten the IC constraint. In contrast with i), when the price of the tree is lower than that of the replicating portfolio, there does not exist an agent holding the replicating portfolio (since holding that portfolio is dominated). Hence arbitrage trades would require the issuance of liabilities, which would tighten the IC constraint (in contrast with ii)). It is striking that, without any exogenous difference in margin constraints or pledgeability between trees and their replicating portfolios, the former are priced below the latter.

Excess return decomposition. The pricing formula (13) leads to a natural decomposition of excess return. Define the risky return on security j as $R_j(\omega) \equiv d_j(\omega)/p_j$ and let the risk-free return be $R_f \equiv 1/\mathbb{E}[M(\omega)]$. Then, standard manipulations of the first order condition (12) show that for almost all securities held by agents of type i:

$$\mathbb{E}[R_j(\omega)] - R_f = -R_f \operatorname{cov}[M(\omega), R_j(\omega)] + R_f \mathbb{E}[A_i(\omega)\delta R_j(\omega)] \tag{15}$$

The first term on the right-hand side of (15) can be interpreted as a risk premium. It is positive if the return on tree j, $R_j(\omega)$, is large for states in which the pricing kernel, $M(\omega)$, is low. It is similar to the standard risk premium associated with consumption betas in frictionless markets but, unlike in the frictionless CCAPM, the pricing kernel $M(\omega)$ mirrors neither aggregate nor individual consumption.

The second term on the right-hand side of (15) is a premium reflecting incentive constraints. This premium is large if non-pledgeable income, $\delta R_j(\omega)$, is large when the incentive compatibility condition is tight.

While equation (15) bears some similarities with the margin-CAPM characterized by Gârleanu and Pedersen (2011), it differs from it in two important ways. First, as shown by the first term of equation (15), the consumption beta in our model is defined relative to the consumption of an unconstrained agent, whose identity differs across states. Second, as shown by the second term of equation (15), the premium associated with financial constraints differs across securities, even though all securities have the same margin requirement, δ. This is because the financial constraints in our model are endogenously state contingent. Correspondingly the premium depends on the covariance between the state-contingent non-pledgeable income

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generated by the security, and the state-contingent shadow price of the constraint.

Basis vs. collateral premium. Equation (13) shows that the tree is priced at a discount relative to the replicating portfolio of Arrow securities, i.e., there is a basis. This could seem to contradict the result obtained in previous literature (see, e.g., Fostel and Geanakoplos, 2008) that equilibrium prices include a collateral premium. There is no contradiction, however, as the premium obtains, both in this paper and in previous literature, relative to a different benchmark than the replicating portfolio.

To see this, consider again the trees held by some agent i. Take the first-order condition (10) with respect to $c_i(\omega)$, multiply by the dividend $d_j(\omega)$ and sum across states to obtain:

$$\mathbb{E}\left[M(\omega)d_j(\omega)\right] = \mathbb{E}\left[\frac{u_i'\left[c_i(\omega)\right]}{\lambda_i}d_j(\omega)\right] + \mathbb{E}\left[A_i(\omega)d_j(\omega)\right]. \tag{16}$$

Substituting (16) into (13) the price of tree j is

$$p_j = \mathbb{E}\left[\frac{u_i'\left[c_i(\omega)\right]}{\lambda_i}d_j(\omega)\right] + \left(\mathbb{E}\left[A_i(\omega)d_j(\omega)\right] - \delta\,\mathbb{E}\left[A_i(\omega)d_j(\omega)\right]\right). \tag{17}$$

This price equation is similar to equation (5) in Fostel and Geanakoplos (2008) or that in Lemma 5.1 in Alvarez and Jermann (2000). The first term on the right-hand side of (17) is similar to what Fostel and Geanakoplos (2008) call "payoff value": it is the expected value of the tree cash flows, evaluated at the marginal utility of the agent holding it. The second term on the right-hand side of (17) is similar to the collateral premium in Fostel and Geanakoplos (2008) (see Lemma 1, page 1230). This premium, however, is reduced by the last term (in factor of δ) which corresponds to the basis whose analysis is one of the contributions of our paper.

5 Two-by-Two

In the previous section, we derived some key implications of our model but did not characterize the equilibrium completely. In particular, we did not derive the endogenous patterns of binding incentive constraints across states and agents, which ultimately determine the equilibrium distribution of trees. In this section,

we provide a full characterization of equilibrium in the natural "two-by-two" case. Namely, we assume that there are two types of agents $i \in \{1, 2\}$, two states, $\omega \in \{\omega_1, \omega_2\}$, and an arbitrary distribution of trees. We further assume that both types of agents, $i \in \{1, 2\}$, have CRRA utility with respective coefficients of relative risk aversion $0 \leq \gamma_1 < \gamma_2 \leq 1$. That is, agent $i = 1$ is more risk tolerant, while agent $i = 2$ is more risk averse.[18]

We normalize the dividend of each tree to one, i.e., $\mathbb{E}[d_j(\omega)] = 1$.[19] Given that there are only two states, any tree can be viewed as a convex combination of two extreme securities: one security that only pays off in state ω_1, and one security that only pays off in state ω_2. Therefore, one can order the trees so that, for any $j \in [0, 1]$,

$$d_j(\omega) = \frac{j}{\pi(\omega_1)} \mathbb{I}_{\{\omega = \omega_1\}} + \frac{1-j}{\pi(\omega_2)} \mathbb{I}_{\{\omega = \omega_2\}}. \tag{18}$$

Notice that, after the normalization $\mathbb{E}[d_j(\omega)] = 1$, in a two-state model the payoff of any security can be represented as (18) for some j.

We label the states so that the aggregate endowment, denoted by $y(\omega) = \int d_j(\omega) \, d\bar{N}_j$, is strictly larger in state ω_2 than in state ω_1:

$$y(\omega_2) = \frac{1}{\pi(\omega_2)} \int (1-j) \, d\bar{N}_j > y(\omega_1) = \frac{1}{\pi(\omega_1)} \int j \, d\bar{N}_j. \tag{19}$$

In other words, ω_1 is the "bad state" while ω_2 is the "good state." The tree $j = \pi(\omega_1)$ is risk free, and so its consumption beta, $\operatorname{cov}[d_j(\omega), y(\omega)] / \operatorname{var}[y(\omega)]$ is zero. Trees with $j < \pi(\omega_1)$ have lower dividend in state ω_1 than in state ω_2, and so have positive consumption beta. The smaller is j, the more positive is the beta. Vice versa, trees with $j > \pi(\omega_1)$ have negative consumption beta. The larger is j, the more negative is the beta. Hence, the parameter j can be interpreted as an index of safety.

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[18]As shown in Proposition C.8 in the appendix, $0 \leq \gamma_1 \leq \gamma_2 \leq 1$ and $\gamma_2 > 0$ imply that the equilibrium consumption allocation is uniquely determined, and the equilibrium prices are uniquely determined up to a multiplicative constant. As shown in Proposition C.7 in appendix, the restriction $\gamma_1 \neq \gamma_2$ is necessary for incentive compatibility to bind in equilibrium.

[19]This is without loss of generality. This merely amounts to divide the dividend in all states by the expected dividend, and simultaneously scaling the tree supply up by the same constant.

5.1 Incentive Feasible Consumption Allocations

In order to characterize the patterns of binding incentive constraints across states and agents, we take a step back and study the set of *incentive feasible* consumption allocations, that is, consumption allocations $(c_i)_{i \in \{1,2\}}$ such that $(c_i, N_i^+)_{i \in \{1,2\}}$ is incentive compatible for some tree allocation $(N_i^+)_{i \in \{1,2\}}$. As will become clear, this turns out to be very useful: indeed, it allows to characterize incentive feasibility as a function of the consumption allocation only, and leave the corresponding feasible allocation of trees implicit.

Focusing on the case in which the consumption share of agent 1 is lower in the bad state than in the good state (which, as shown below, is the case in equilibrium), our first main result is the following:

Lemma 2 *Consider a feasible consumption allocation such that $c_1(\omega_1)/y(\omega_1) < c_1(\omega_2)/y(\omega_2)$. Then, c is incentive feasible if and only if there exists $k \in [0, 1]$ and $(\Delta N_1, \Delta N_2) \geq 0$, $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$, such that:*

$$c_1(\omega_1) \geq \delta \int_{j \in [0,k)} d_j(\omega_1) d\bar{N}_j + \delta d_k(\omega_1) \Delta N_1 \tag{20}$$

$$c_2(\omega_2) \geq \delta \int_{j \in (k,1]} d_j(\omega_2) d\bar{N}_j + \delta d_k(\omega_2) \Delta N_2. \tag{21}$$

Equation (20) is the incentive compatibility condition of agent $i = 1$ in state ω_1 when he holds all trees riskier than k ($j < k$), and, if there is an atom at k in the distribution of trees, a mass ΔN_1 of that atom. Similarly, equation (21) is the incentive compatibility condition of agent $i = 2$ when he holds all trees $j > k$, plus a mass ΔN_2 of the atom at k, if there is one. The lemma thus states that a consumption allocation such that $c_1(\omega_1)/y(\omega_1) < c_1(\omega_2)/y(\omega_2)$ is incentive feasible if and only if it can be implemented by allocating riskier trees to the more risk tolerant agent 1 and safer trees to the more risk averse agent 2 without violating their incentive compatibility constraints in states ω_1 and ω_2, respectively. The intuition for this result can be grasped from the following two observations.

The first observation is that, since her consumption share is smaller in ω_1 than in ω_2, agent $i = 1$ tends to have incentive problems in state ω_1. To understand why, imagine that agent $i = 1$ purchases a fraction of the market portfolio equal to her average consumption share across states. In order to implement her consumption plan $c_1(\omega)$ while holding this portfolio, agent $i = 1$ has to sell Arrow securities that pay off in

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state ω_1, and purchase Arrow securities that pay off in state ω_2. Hence, agent $i = 1$ only has a liability in state ω_1, so that her incentive compatibility constraint can bind only in that state. Vice versa, agent $i = 2$ faces incentives problems only in state ω_2. The lemma states that only two (equations (20) and (21)) out of the four incentive compatibility constraints matter.

The second observation is that, to mitigate these incentive problems, it is best to allocate agent $i = 1$ a portfolio of trees with low payoff in state ω_1. This minimizes the amount this agent can renegotiate in the state in which his incentive constraint binds. Symmetrically, it is best to allocate agent $i = 2$ a portfolio of trees with low payoff in state ω_2. By market clearing an equivalent way to grasp the intuition for this result is the following: Allocating to the more risk-averse agents trees with relatively high payoff in the bad state reduces the reliance of that agent on insurance sold by the more risk tolerant agent. This, in turn, relaxes incentives constraints.

The lemma is illustrated in the Edgeworth box in Figure 1. The consumption of agent $i = 1$ in state ω_1 is on the x-axis, and his consumption in state ω_2 is on the y-axis. The curves above and below the diagonal are the boundaries of the incentive feasible set. In line with the lemma, focus on the area above the diagonal, where the consumption share of agent 1 is larger in the good state, ω_2, than in the bad state ω_1. All the area between the boundary of the incentive set and the diagonal is incentive compatible.

One sees in the figure that any allocation which gives sufficiently small consumption to one of the agents is incentive feasible. For example, if the consumption of agent $i = 1$ is sufficiently small then the consumption of agent $i = 2$ is almost equal to the aggregate endowment. As long as $\delta < 1$, this allocation can be made incentive feasible by allocating all the trees to agent $i = 2$. In equilibrium, agent $i = 1$ sells all his trees to agent $i = 2$, and agent $i = 2$ issues a liability corresponding to agent $i = 1$ consumption. This is feasible since $\delta < 1$ gives agent $i = 2$ some borrowing capacity.

Figure 1 also compares the incentive sets for one tree and for many trees (keeping aggregate output in each state constant.) The dashed line is the boundary of the incentive-feasible set when there is just one tree in strictly positive supply.[20] The solid line is the boundary when there are many trees.[21] The figure

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[20]In that case, the distribution $\bar{N}$ has just one atom. If we normalize this atom to one for simplicity, then in the Edgeworth box the boundary is the curve parameterized by $\Delta N_1 \in [0, 1]$, with cartesian coordinates $c_1(\omega_1) = \delta d(\omega_1) \Delta N_1$ and $c_1(\omega_2) = y(\omega_2) - c_2(\omega_2) = d(\omega_2) [1 - \delta + \delta \Delta N_1]$

[21]In that case we assume no atom, so the boundary is the curve parameterized by $k \in [0, 1]$, with cartesian coordinates $c_1(\omega_1) = \delta \int_0^k d_j(\omega_1) \, d\bar{N}_j$ and $c_1(\omega_2) = y(\omega_2) - c_2(\omega_2) = \int_0^1 d_j(\omega_2) \, d\bar{N}_j - \delta \int_k^1 d_j(\omega_2) d\bar{N}_j$.
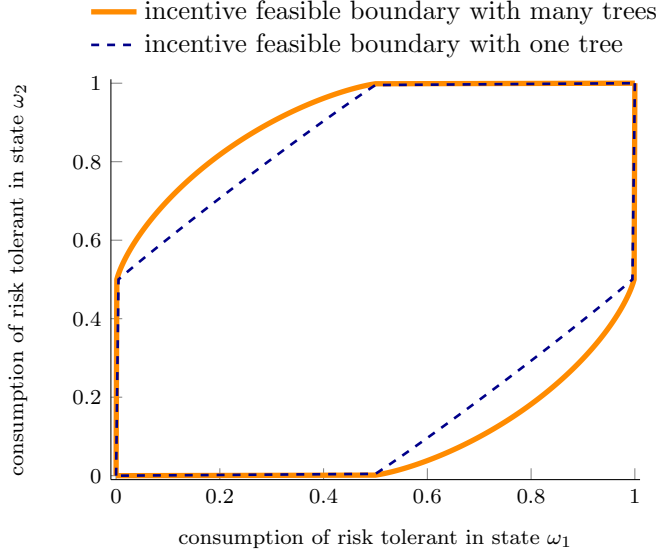
Figure 1: The set of incentive feasible consumption allocations in an Edgeworth box, when $\pi(\omega_1) = 0.1$ and $\delta = 0.5$. In the many-trees case, tree supplies are distributed according to a beta distribution with parameters $a = b = 5$. In the one-tree case, there is just one tree equal to the market portfolio of the many-trees case.

illustrates that the incentive-feasible set is smaller with one tree than with many trees. Indeed, with many trees, one can replicate one-tree allocations by allocating agents shares in the market portfolio.

5.2 Equilibrium Allocations

In order to characterize equilibrium allocations, we rely on their efficiency properties. Let $(c_i, N_i^+)_{i \in \{1,2\}}$ denote the equilibrium allocation. As shown in Proposition 2, $(c_i, N_i^+)_{i \in \{1,2\}}$ is constrained Pareto efficient. That is, $(c_i, N_i^+)_{i \in \{1,2\}}$ solves an *incentive-constrained* planner's problem, i.e., there exist weights $(\alpha_1, \alpha_2) \in (0,1)^2$, $\alpha_1 + \alpha_2 = 1$, such that c maximizes $\sum_{i \in I} \alpha_i U_i(c_i)$ with respect to feasible allocations satisfying the incentive compatibility conditions. Let $c^\star$ denote the solution of the corresponding *unconstrained* planner's problem. That is, $c^\star$ maximizes the same welfare function, with the same weights (α_1, α_2), with respect to feasible allocations, but without imposing the incentive compatibility conditions.

Lemma 3 *If $(\alpha_1, \alpha_2) > 0$, then the solutions of the unconstrained and incentive-constrained planner's problems both are such that $c_1^\star(\omega_1)/y(\omega_1) < c_1^\star(\omega_2)/y(\omega_2)$ and $c_1(\omega_1)/y(\omega_1) < c_1(\omega_2)/y(\omega_2)$.*

The lemma states that the more risk tolerant agent, $i = 1$, receives a lower share of aggregate consumption in the low state than in the high state (as in the first best). Since consumption shares add up to one across

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agents, it follows that the risk averse agent, $i = 2$, enjoys a higher share of aggregate consumption in the low than in the high state. Intuitively, a consumption allocation which delivers a constant consumption share in both states to both agents is always strictly incentive feasible since it lies on the diagonal of the Edgeworth box of Figure 1. But the risk tolerant cares relatively less about the low state, ω_1, and relatively more about the high state, ω_2. Hence, social welfare increases strictly if the risk tolerant agent, $i = 1$ insures the more risk averse agent by letting $i = 2$ have a larger share of aggregate consumption in the bad state.

Lemma 3 states that the planner always finds it optimal to pick consumption allocations above the diagonal of the Edgeworth box. Therefore, the relevant incentive constraint is the upper boundary of the incentive feasible set in Figure 1. Using Lemma 2, we then obtain our next proposition:

Proposition 6 *If $c \neq c^\star$, then the incentive compatibility constraint of agent $i = 1$ binds in state ω_1, while the incentive compatibility constraint of agent $i = 2$ binds in state ω_2. Moreover, there exists $k \in [0, 1]$ and $(\Delta N_1, \Delta N_2) \geq 0$, $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$, such that agent $i = 1$ holds all trees $j < k$, i.e.,*

$$N_1^+ = \bar{N} \, \mathbb{I}_{\{j<k\}} + \Delta N_1 \mathbb{I}_{\{j=k\}}$$

and agent $i = 2$ holds all trees $j > k$, i.e., $N_2^+ = \bar{N} \, \mathbb{I}_{\{j>k\}} + \Delta N_2 \mathbb{I}_{\{j=k\}}$.

Lemma 2 stated that a consumption allocation was incentive feasible if and only if it could be implemented by allocating the riskier trees to the more risk tolerant agent and the safer trees to the more risk averse agent. In line with that result, Proposition 6 states that, in equilibrium, the binding incentive constraints of the more risk tolerant agent in the bad state, and the more risk averse agent in the good state, pin down such an allocation. This equilibrium allocation can be interpreted in terms of segmentation, as different classes of investors hold different types of trees.

The proposition is illustrated in Figure 2. In the figure, the "constrained Pareto set" and the "unconstrained Pareto set" are, respectively, the set of consumption allocations obtained by solving the incentive-constrained and the unconstrained Planner's problem for all possible weights $(\alpha_1, \alpha_2) \in [0, 1]^2$, $\alpha_1 + \alpha_2 = 1$. The incentive-constrained Pareto set coincides with the unconstrained Pareto set when the latter lies below the upper boundary of the incentive-feasible set. Otherwise, the incentive-constrained Pareto set coincides
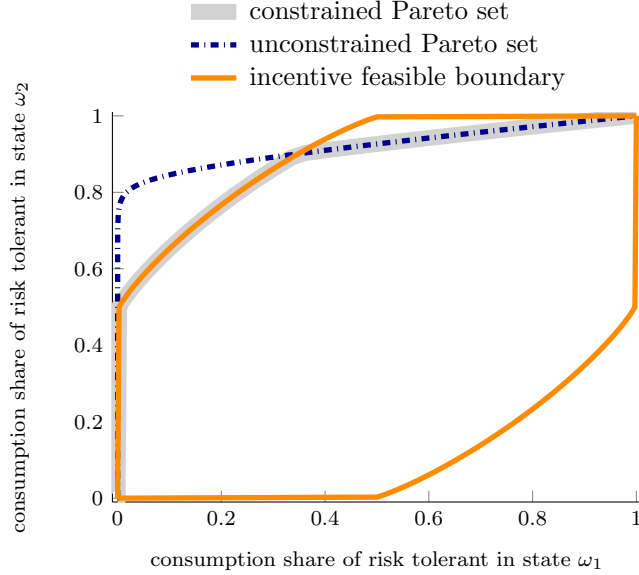
Figure 2: The set of incentive-feasible and the set of incentive-constrained Pareto allocation, when $\pi(\omega_1) = 0.9$, $\gamma_1 = 0.1$, $\gamma_2 = 1$, and $\delta = 0.5$. In the many-trees case, tree supplies are distributed according to a beta distribution with parameters $a = b = 15$. In the one-tree case, there is just one tree equal to the market portfolio of the many-trees case.

with the IC boundary. As α_1/α_2 increases, then the constrained Pareto efficient allocation moves monotonically to the northeast of the Edgeworth box of Figure 1.

Next, we translate the above discussion into equilibrium comparative statics. To do so, all we need is to study the mapping between the exogenous endowments $(\bar{n}_1, \bar{n}_2)$ and their corresponding *endogenous Pareto weights*, that is, the weights (α_1, α_2) such that the equilibrium allocation given endowments $(\bar{n}_1, \bar{n}_2)$ is the solution of the incentive-constrained Planner's problem. The existence of endogenous Pareto weights is immediate from Propositions 2 and 3. We obtain:

Proposition 7 *The ratio of endogenous Pareto weights, α_1/α_2, is strictly increasing in the ratio of initial endowment $\bar{n}_1/\bar{n}_2$.*

Thus, while Figure 2 reveals that incentive compatibility does not matter for extreme values of α_1/α_2, Proposition 7 enables one to restate this observation in terms of the distribution of initial endowments. When this distribution is very unequal, as agents $i = 1$ are initially endowed with a very large fraction of the market portfolio ($\bar{n}_1/\bar{n}_2$ large), or agents $i = 2$ are initially endowed with a very large fraction of the market portfolio ($\bar{n}_1/\bar{n}_2$ low), there is little scope for risk sharing between the two types of agents. Thus, even the unconstrained equilibrium involves little trading, so that incentive constraints do not bind (as is

the case in Figure 2 in the north east and the south west of the Edgeworth box). In contrast, when the distribution of initial endowments is more equal ($\bar{n}_1$ close to $\bar{n}_2$) the scope for risk sharing is large. In that case the incentive-constrained equilibrium allocation differs significantly from its unconstrained counterpart (as is the case in Figure 2 in the north west of the Edgeworth box). Correspondingly, the basis is zero when the distribution of endowments is very unequal, while it can be strictly positive when initial endowments are equally distributed.

5.3 Relative Supply Effects

In our model, the relative supply of trees (i.e., the distribution $\bar{N}_j$) determines equilibrium outcomes, by changing the shape of the incentive feasible set. Changing the relative supplies of trees changes equilibrium outcomes even if it does not change aggregate output and pledgeable income in each state, nor the span of the trees' payoff matrix. This is in sharp contrast with the perfect and complete markets case, in which the set of feasible allocations is not affected by the way the aggregate output is split across trees.

As an illustration, compare an economy with just one tree (the "market portfolio") to another economy in which there are two trees, one paying off only in the good state, the other paying off only in the bad state. To reason ceteris paribus, the aggregate output is the same in each state in the two economies. If incentive problems are severe, because $\delta \simeq 1$, then, when there is only one tree, incentive constraints are more likely to bind in equilibrium. In contrast, when there are two trees, each paying off only in one state, all agents can attain their $\delta = 0$ equilibrium consumption just by holding trees. Hence incentive constraints do not bind, even if $\delta \simeq 1$.[22]

When there is a single tree, its consumption β is equal to one, while with two trees one has a lower β and the other a higher one. The comparison between the single tree economy and its two tree counterpart suggests that increasing the dispersion of βs relaxes incentive constraints. To make this point, first note that in our simple two-state case, the consumption β of tree j is affine in j with coefficients that only depend on the values and probabilities of aggregate dividends. Therefore, the dispersion of βs is increasing in the dispersion of js.

To model an increase in the dispersion of the distribution of trees, from $\bar{N}$ to $\bar{N}^*$, it is natural to consider

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[22]A proof of this and related results is provided in the appendix in Proposition C.7.

a mean preserving spread, that is, a decrease in the sense of second order stochastic dominance:

$$\int_0^k \bar{N}_j^* \geq \int_0^k \bar{N}_j, \qquad \text{for all } k \in [0,1], \tag{22}$$

preserving the aggregate output in each state. From equation (19), one sees that aggregate output is preserved in each state if and only if:

$$\int_0^1 j \, d\bar{N}_j^* = \int_0^1 j \, d\bar{N}_j \tag{23}$$

$$\bar{N}_1^* = \bar{N}_1. \tag{24}$$

In this context, we obtain the following proposition:

Proposition 8 *Consider two tree supply distributions $\bar{N}$ and $\bar{N}^*$, such that $\bar{N}^*$ is more dispersed than $\bar{N}$ in the sense that (22), (23) and (24) hold. If an allocation is incentive feasible for $\bar{N}$, then it is incentive feasible for $\bar{N}^*$.*

Proposition 8 states that the set of incentive feasible allocations expands as the tree supply distribution becomes more dispersed, i.e., the dispersion of βs increases. The proposition enables one to revisit and generalise the intuition obtained by comparing the single tree and two-tree economies: a mean-preserving spread implies that the supply distribution $\bar{N}^*$ puts more weight on the riskiest trees, and also on the safest trees, than the distribution $\bar{N}$. This implies that the set of trees held by the more risk tolerant agent is overall riskier, and correspondingly yields smaller dividends in the bad state, which relaxes the incentive compatibility condition of this agent. Symmetrically, by putting more weight on the safest trees, the supply distribution $\bar{N}^*$ leads to an overall safer set of trees held by the more risk averse agent. This reduces the dividends of these trees in the good state, which relaxes the incentive constraint of the more risk averse agent.

The tightening of incentive constraints, induced by a decrease in the dispersion of consumption betas, affects equilibrium pricing. This is illustrated in Figure 3, which plots the equilibrium expected return on the market portfolio for different dispersions of consumption betas, keeping aggregate output in each
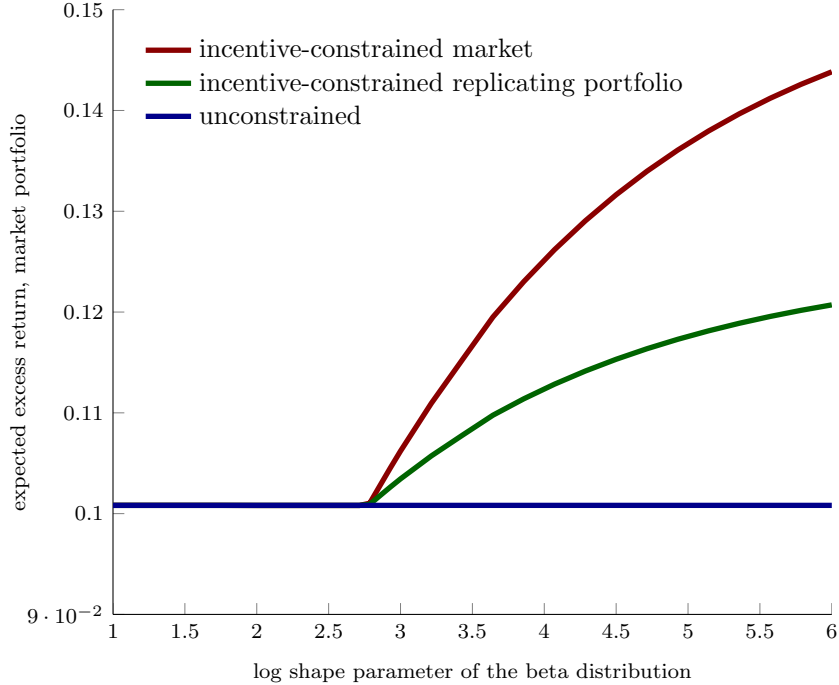
Figure 3: The expected excess return on the market portfolio as the dispersion of security beta decreases. Tree supplies are distributed according to a beta distribution with shape parameters $a = b$ varying, in logs, from 1 to 6.

state constant. We consider symmetric beta distributions, and let the log of their shape parameter increase from 1 to 6, so that their dispersion decreases while their mean remains constant. The figure shows that, as the dispersion of consumption betas decreases, the expected return of the portfolio of Arrow securities replicating the market increases, reflecting poorer allocation of risks in the economy. Moreover, as betas get less dispersed, the basis increases, reflecting the increased shadow costs of incentive constraints. In contrast, by construction, when there are no incentive problems the expected return on the market portfolio is not affected by changes in the dispersion of betas.

5.4   The Cross Section of Bases

Focus on the case in which incentive compatibility constraints bind. By Proposition 6, there exists a threshold k, such that the more risk tolerant agent holds trees $j < k$, while the more risk averse agent holds trees $j > k$. In this context, from the first-order condition (12), the basis on tree $j < k$ held by the risk tolerant

agent, $i = 1$, is

$$\sum_{\omega \in \Omega} q(\omega) d_j(\omega) - p_j = \delta \frac{\mu_1(\omega_1)}{\lambda_1} d_j(\omega_1).$$

Since only relative prices are pinned down, we express this basis in relative prices, and choose as normalizing factor (or numeraire) the price of the riskless bond, $q(\omega_1) + q(\omega_2)$. For tree $j < k$ the normalised basis is thus

$$\Delta_j = \left[\frac{\mu_1(\omega_1)/\lambda_1}{q(\omega_1) + q(\omega_2)} \right] \times \left[\delta \, d_j(\omega_1) \right]. \tag{25}$$

The first term in the right-hand side of equation (25) is constant across all trees held by agent 1, and measures, intuitively, the tightness of the incentive constraint of agent 1. The second term is equal to the non-pledgeable cash flow of the tree in state ω_1 in which the agent holding it is constrained. This term, and correspondingly the basis, is higher for trees with a relatively large payoff in the bad state and a relatively low payoff in the high state, that is, trees with a lower consumption beta. The intuition is that the risk tolerant agent sells insurance against the bad state to the risk averse agent. However, the incentive compatibility constraint limits the amount of insurance she can sell. Since the consumption of the risk tolerant agent is low in the bad state, renegotiating is tempting. It implies that the shadow cost of holding a tree is higher for trees paying relatively more in the bad state, i.e., for trees with a lower consumption beta. Remember however that the risk tolerant agent holds trees with a high beta. Therefore, among trees with a high consumption beta, trees with a moderately high beta have a larger basis than trees with a very high beta.

Consider now trees $j > k$ held by agent 2. Following the same reasoning as before, the basis equals

$$\Delta_j = \left[\frac{\mu_2(\omega_2)/\lambda_2}{q(\omega_1) + q(\omega_2)} \right] \times \left[\delta \, d_j(\omega_2) \right]. \tag{26}$$

Equation (26) implies that, among trees held by the risk averse agent, $i = 2$, the basis is larger for trees with a relatively large payoff in the good state and a relatively low payoff in the bad state, that is, with a higher consumption beta. The intuition is symmetric to the one above. The risk averse agent would like to sell consumption to the risk tolerant agent in the good state, but it is tempting for the risk averse agent to renegotiate in the good state. Thus, the shadow cost of holding a tree is higher for tree with a relatively high payoff in the good state, that is, for trees with a higher consumption beta. The risk averse agent holds
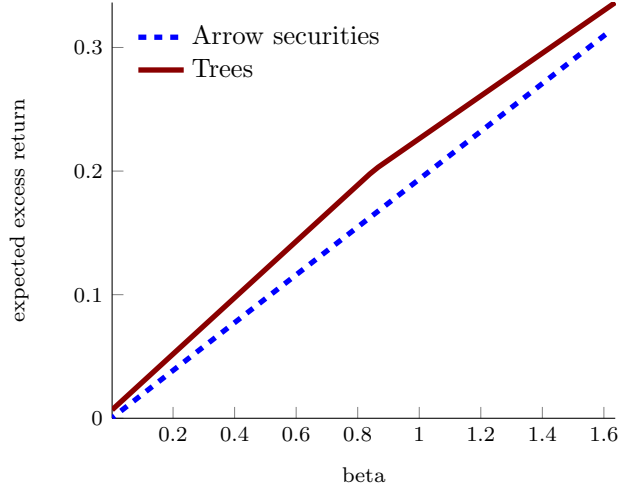
Figure 4: The Modified Security Market Line

trees with a low consumption beta. Therefore, among trees with a low beta, those with a moderately low beta have a larger basis than trees with a very low beta. Putting things together, we conclude that:

Proposition 9 *Suppose all trees are in strictly positive supply. Then, the basis induced by incentive constraints is an inverse U-shape function of the consumption beta of the tree.*

The restriction that all trees are in positive supply ensures that prices are uniquely determined. Intuitively, the proposition means that, after adjusting for risk, trees with either a low or a large consumption beta will tend to have a high price, and a low return. This is illustrated in Figure 4. The figure shows the security market line (SML) in our environment, which we derive explicitly in Proposition E.1 in the appendix. Since trees are held by agents who value them most, the SML is the minimum between the SML obtained from agent $i = 1$'s valuation, and that derived from agent $i = 2$'s valuation. The kink in the figure occurs at tree k, for which ownership switches from agent 1 to agent 2. The figure illustrates that, because the basis is inverse-U shaped in β, the SML is flatter at the top and steeper at the bottom, in line with Black (1972), and recent evidence in Frazzini and Pedersen (2014) and Hong and Sraer (2016).

Finally, we re-cast our findings in an option-pricing context. Recall that, following a standard binomial option-pricing argument, a bond is replicated by a portfolio made up of a stock and of a put option. Now consider a sufficiently risky stock (low j or high beta) and an out-of-the-money put, interpreted as an Arrow security with positive payoff in the low state, and zero payoff in the high state. Then the stock is held by the

risk tolerant agent while the put option, which only pays off in the low state, must be held by the risk averse agent. It thus follows from Proposition 5 that the bond is priced strictly below the replicating portfolio: that is, in line with empirical evidence, out-of-the-money puts appear too expensive relative to standard arbitrage relationships. A symmetric reasoning reveals that, if risk-free bonds are held by risk averse agents, then out-of-the-money calls also appear too expensive. According to our model, these price discrepancies arise because of limits to arbitrage induced by incentive constraints. For example, to take advantage of the price discrepancy, the risk tolerant agent would need to sell the high beta stock, and sell more of the put, which he cannot do because of the incentive constraint.

5.5   Response to Shocks on Incentive Problems

To model shocks on incentives, fix a tree $\ell < k$ and consider a small increase in δ for tree ℓ and possibly nearby trees. Formally, assume $\delta_j = \delta + \varepsilon \phi_j$ for some continuous function ϕ_j strictly positive near ℓ, and zero everywhere else.[23] The shock worsens incentive problems for tree ℓ (and neighbouring trees). What is the effect of this shock on allocations?

Proposition 10 *Assume that the cumulative distribution of trees is continuous and strictly increasing, that $c \neq c^\star$, and that $k \in (0, 1)$. The ε shock shrinks the set of trees held by agent 1: $k(\varepsilon) < k(0)$ for small $\varepsilon > 0$.*

When agent 1 becomes slightly worse at pledging a tree he already holds, the shadow value of his incentive-compatibility constraint increases, which makes it more costly for the more risk tolerant agent 1 to hold any tree. Thus, in equilibrium, the set of trees $[0, k)$ held by agent 1 shrinks. This means that agent $i = 1$ sells the safest trees in his portfolio, while keeping the riskiest ones. This might sound counter-intuitive if one expected that, when an agent's incentive problems become more severe, he should sell his riskiest trees. The result arises because, even after the shock, agent $i = 1$ is still in a better position to hold very risky trees than the more risk averse agent $i = 2$. Thus, as the shock reduces agent $i = 1$'s ability to hold trees overall, he divests those for which his comparative advantage is the lowest.

Now turn to the effect of the shock on prices. Equations (25) and (26) imply that when agent $i = 1$ becomes a worse pledger for tree ℓ, the basis of tree ℓ increases relative to the bases of other trees j held by

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[23]All of our results extend to this case. In fact, our proofs in the appendix cover the case of δ which are continuously varying across agents and tree types.

$i = 1$ (that have not been directly hit by the shock), which themselves increase relative to the bases of trees j' held by $i = 2$. That is

$$\frac{\Delta_\ell(\varepsilon)}{\Delta_\ell(0)} > \frac{\Delta_j(\varepsilon)}{\Delta_j(0)} > \frac{\Delta_{j'}(\varepsilon)}{\Delta_{j'}(0)}$$

Thus, co-movement in the basis induced by incentive constraints is stronger among trees held by the same type of agents.

6   Conclusion

We introduce incentive compatibility constraints, limiting the pledgeability of collateral, in an otherwise standard general equilibrium model. In each state, agents cannot pledge more than a fraction of the payoff from their holdings in that state. Hence, although a complete set of Arrow securities are available for trade, limited collateral pledgeablity drives risk-sharing below its first best counterpart. Thus, markets are endogenously incomplete.

To cope with incentive constraints, relatively risk averse agents hold low beta securities, while relatively risk tolerant agents hold high beta securities. This reflects the risk tolerant agents' comparative advantage at holding risky securities. Correspondingly, the market is endogenously segmented.

When hit by an adverse shock on incentives, relatively risk tolerant agents sell their safest securities, not their riskiest ones. This is because, while the shock reduces their ability to hold all securities, it does not eliminate their comparative advantage at holding the riskiest ones.

Incentive compatibility constraints generate a basis between the prices of risky securities and those of replicating portfolios of derivatives (in spite of the fact that risky securities and derivatives are equally imperfectly pledgeable). The basis always goes in the same direction: the price of risky securities is below that of replicating derivative portfolios. Arbitraging the basis would imply buying the "cheap" risky securities, and selling the "expensive" derivatives, but the latter sale is ruled out by incentive compatibility. The structure of the basis is such that equilibrium expected excess returns are concave in consumption betas, in line with empirical findings. Moreover an increase in the dispersion of consumption betas relaxes incentive constraints, which reduces the basis. The latter two results are related. The concavity of the security

market line reflects that securities with extreme betas are more valuable because they enable agents to share risk without creating incentive problems. An increase in the dispersion of consumption betas increases the prevalence of securities with extreme betas, which relaxes incentive constraints in equilibrium.

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Online Appendix

A Derivation of the incentive compatibility condition (1)

First, we state formally the assumptions of the renegotiation game. If agent i renegotiates in state ω, it makes a take-or-leave-it offer to repay X instead of $\int d_j(\omega) \, dN_{ij}^- + a_i^-(\omega)$. A third party (e.g., a bankruptcy court) whose objective is to maximize the payment to agent i's claimholders, decides whether to accept the offer or not. If the offer is accepted, claimholders obtain X and the agent gets the value of its long positions in trees and Arrow securities minus X. We need to specify the payoffs of the agent and the claimholders when the offer is rejected and the agent's assets are seized. Because assets are imperfectly seizable, claimholders obtain fraction $1 - \delta$ of the value of the agent's long positions. Moreover, we assume the agent gets a fraction $\kappa \in [0,1]$ of the payoffs of the fraction δ of the agent's long positions which creditors cannot seized. Hence, κ determines whether the part of collateral value that creditors cannot seize is a deadweight loss (if $\kappa = 0$) or diverted by the debtor ($\kappa = 1$) or any case in between. As will be clear below, the incentive compatibility condition does not depend on the value of κ.

Now, we determine the incentive compatibility condition (1) by requiring that liabilities are renegotiation proof. If the agent does not renegotiate, its payoff is assets minus repayment:

$$\left[\int d_j(\omega) \, dN_{ij}^+ + a_i^+(\omega)\right] - \left[\int d_j(\omega) \, dN_{ij}^- + a_i^-(\omega)\right]. \tag{27}$$

If the agent renegotiates and offers to repay X, the offer is accepted if and only if the offered repayment is no less than the fraction $1 - \delta$ of collateral value:

$$X \geq (1 - \delta)\left[\int d_j(\omega) \, dN_{ij}^+ + a_i^+(\omega)\right]. \tag{28}$$

Clearly, conditional on making an offer that is accepted, it is optimal for the agent to offer the smallest possible X that satisfies (28), $X = (1 - \delta)\left[\int d_j(\omega) \, dN_{ij}^+ + a_i^+(\omega)\right]$, leaving the agent with payoff:

$$\delta \left[\int d_j(\omega) \, dN_{ij}^+ + a_i^+(\omega)\right]. \tag{29}$$

If the offer is rejected, then the payoff is $\kappa \, \delta \int d_j(\omega) \left[dN_{ij}^+ + a_i^+(\omega)\right]$. Therefore, conditional on renegotiating, it is optimal for the agent to make an offer that is accepted. Combining (27) and (29), the agent does not renegotiate if and only if (1) holds.

B Proofs of the results in Section 3

B.1 Proof of Lemma 1

The proof follows the two steps outlined in the text. First, we show that the candidate equilibrium allocation $(\hat{c}_i, \hat{N}_i, \hat{a}_i)_{i \in I}$ clears markets. Second, we show that the plan $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ satisfies the incentive constraints of each agent. Third, we show that the plan $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ is budget feasible for all agents given prices (p, q). Taken together, these results show that the allocation $(\hat{c}_i, \hat{N}_i, \hat{a}_i)_i \in I$ is the basis of a financial-market equilibrium given prices (p, q).

Step 1: market clearing. As indicated in the text, we let $\hat{c}_i = c_i$. We also set $\hat{N}_i^- = 0$ and we scale down long tree positions so that tree markets clear. Formally, the existence of an appropriate scaling factor, $1 - \theta_j$, follows from an application of the Radon-Nikodym Theorem (see, e.g., Royden and Fitzpatrick, 2010, page 382). Indeed, the market clearing condition, $\sum_{i \in I} N_i^+ = \bar{N} + \sum_{i \in I} N_i^-$, implies that $\bar{N}$ is absolutely continuous with respect to $\sum_{i \in I} N_i^+$. It thus follows from the Radon-Nikodym Theorem that there exists a measurable function $j \mapsto \theta_j \leq 1$ such that

$$d\bar{N}_j = \sum_{i \in I} (1 - \theta_j) \, dN_{ij}^+. \tag{30}$$

Hence, if long tree positions are scaled down by $1 - \theta_j$, i.e. if we let $d\hat{N}_{ij}^+ \equiv (1 - \theta_j) \, dN_{ij}^+$, then the portfolios of long and short positions $(\hat{N}_i^+, \hat{N}_i^-)_{i \in I}$ clear the market for trees. Finally, we consider the Arrow securities positions

$$\hat{a}_i^+(\omega) = \max \left\{ a_i^+(\omega) - a_i^-(\omega) + \int d_j(\omega) \left[\theta_j \, dN_{ij}^+ - dN_{ij}^- \right], 0 \right\} \tag{31}$$

$$\hat{a}_i^-(\omega) = \max \left\{ a_i^-(\omega) - a_i^+(\omega) + \int d_j(\omega) \left[dN_{ij}^- - \theta_j \, dN_{ij}^+ \right], 0 \right\}. \tag{32}$$

By construction, these satisfy

$$\hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) + \int d_j(\omega) \left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^- \right] = a_i^+(\omega) - a_i^-(\omega) + \int d_j(\omega) \left[dN_{ij}^+ - dN_{ij}^- \right], \tag{33}$$

for all $\omega \in \Omega$, that is, the net security position (Arrow securities and trees) remain the same in all states. Using the market clearing for trees in the original allocation $(c_i, a_i, N_i)_{i \in I}$, (30) we obtain that

$$\sum_{i \in I} \left[dN_{ij}^+ - dN_{ij}^- \right] = \sum_{i \in I} (1 - \theta_j) \, dN_{ij}^+ \implies \sum_{i \in I} \left[\theta_j dN_{ij}^+ - dN_{ij}^- \right] = 0.$$

Combining this with the definition of $\hat{a}_i$, and using the market clearing for Arrow securities in the original equilibrium allocation, $(c_i, a_i, N_i)_{i \in I}$, we obtain market clearing for Arrow securities for the modified allocation, $(\hat{c}_i, \hat{a}_i, \hat{N}_i)_{i \in I}$.

Step 2: incentive compatibility. Next, we verify that all incentive constraints hold. To that end, notice that (1) can be rewritten:

$$\delta \left[\hat{a}_i^+(\omega) + \int d_j(\omega) \, d\hat{N}_{ij}^+ \right] \leq \hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) + \int d_j(\omega) \left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^- \right]. \tag{34}$$

Plugging in the definition of $\hat{a}_i^+(\omega)$ and $\hat{a}_i^-(\omega)$, in equations (31)-(32):

$$\begin{aligned}
\delta \left[\hat{a}_i^+(\omega) + \int d_j(\omega) \, d\hat{N}_{ij}^+ \right] &= \delta \left[\max \left\{ a_i^+(\omega) - a_i^-(\omega) + \int d_j(\omega) \left[\theta_j \, dN_{ij}^+ - dN_{ij}^- \right], 0 \right\} + \int d_j(\omega) (1 - \theta_j) dN_{ij}^+ \right] \\
&\leq \delta \left[a_i^+(\omega) + \int d_j(\omega) \, dN_{ij}^+ \right] \\
&\leq a_i^+(\omega) - a_i^-(\omega) + \int d_j(\omega) \left[dN_{ij}^+ - dN_{ij}^- \right] = \hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) + \int d_j(\omega) \left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^- \right],
\end{aligned}$$

where: the equality on the first line follows by definition of the candidate equilibrium allocation, $(\hat{c}_i, \hat{a}_i, \hat{N}_i)$; the inequality on the second line follows because, for any $(a, b) \in \mathbb{R}^2$, $\max\{a + b, 0\} \leq \max\{a, 0\} + \max\{b, 0\}$; the inequality on the third line follows because the original equilibrium allocation satisfy all incentive constraint and because incentive constraints can be written as in (34); and the equality on the third line follows because the

candidate equilibrium allocation keeps all net security positions the same as in the original equilibrium allocation, as shown in (33).

Step 3: budget feasibility. We first establish that agents do not find optimal to long (short) trees that are priced strictly above (below) replicating portfolio of Arrow securities. Formally, we let

$$
\begin{aligned}
J_0^+ &\equiv \{j \in [0,1] : p_j > \sum_{\omega \in \Omega} q(\omega) d_j(\omega)\} \\
J_0^- &\equiv \{j \in [0,1] : p_j < \sum_{\omega \in \Omega} q(\omega) d_j(\omega)\},
\end{aligned}
$$

and we show that $N_i^+(J_0^+) = N_i^-(J_0^-) = 0$. We only state the proof that $N_i^+(J_0^+) = 0$, as the proof that $N_i^-(J_0^-) = 0$ is symmetric. Let us assume, towards a contradiction, that $N_i^+(J_0^+) > 0$. Then, consider an alternative plan that keeps the short positions the same, scales down the portfolio of over-priced trees, $j \in J_0^+$, by a factor $1 - \lambda < 1$, and replace these trees by corresponding positions in their replicating portfolios of Arrow securities. Since these trees are over-priced, this generates a profit at time zero which can be used to increase consumption in each state. Formally, the alternative plan is $(\tilde{c}_i, \tilde{a}_i, \tilde{N}_i)$ such that $\tilde{a}_i^- = a_i^-$, $\tilde{N}_i^- = N_i^-$, $\tilde{a}_i^+(\omega) = a_i^+(\omega) + \lambda \int_{j \in J_0^+} d_j(\omega) \, dN_{ij}^+ + \Delta$, $d\tilde{N}_{ij}^+ = dN_{ij}^+ - \lambda \mathbb{I}_{\{j \in J_0^+\}} dN_{ij}^+$, where $\Delta > 0$ is chosen so that $\sum_{\omega \in \Omega} q(\omega)\Delta = \lambda \int_{j \in J_0^+} \left[p_j - \sum_{\omega \in \Omega} q(\omega) d_j(\omega) \right] dN_{ij}^+$. One easily sees that the incentive constraint (1) is relaxed, since, on the right-hand side, the total long security position is increased by Δ in all states while, on the left-hand side, the total short security position stays the same. Clearly, the alternative plan increases the agent's utility strictly, which contradicts optimality.

We now show that the candidate equilibrium allocation $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ constructed above is budget feasible at $t = 0$ for all $i \in I$. The left-hand side of the time-zero budget constraint evaluated at $(\hat{N}_i, \hat{a}_i)$ is:

$$
\begin{aligned}
&\sum_{\omega \in \Omega} q(\omega) \left[\hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) \right] + \int p_j \left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^- \right] \\
= &\sum_{\omega \in \Omega} q(\omega) \left[a_i^+(\omega) - a_i^-(\omega) \right] + \int p_j \left[dN_{ij}^+ - dN_{ij}^- \right] + \int \left[\sum_{\omega \in \Omega} q(\omega) d_j(\omega) - p_j \right] \left[\theta_j dN_{ij}^+ - dN_{ij}^- \right],
\end{aligned} \tag{35}
$$

where we used the expressions for $\hat{N}_i$ and $\hat{a}_i$. Notice that the first two terms in equation (35) correspond to the left-hand side of the time-zero budget constraint for the original plan (c_i, a_i, N_i). Hence, the time-zero budget constraint holds for the candidate plan $(\hat{c}_i, \hat{a}_i, \hat{N}_i)$ if the third term in equation (35) is negative. Using the definition of J_0^+ and J_0^-, as well as our preliminary result that $N_i^+(J_0^+) = 0$ and $N_i^-(J_0^-) = 0$, we obtain that this term can be written

$$
-\int_{j \in J_0^+} \left[\sum_{\omega \in \Omega} q(\omega) d_j(\omega) - p_j \right] dN_{ij}^- + \int_{j \in J_0^-} \left[\sum_{\omega \in \Omega} q(\omega) d_j(\omega) - p_j \right] \theta_j dN_{ij}^+. \tag{36}
$$

Now we argue that both terms must be equal to zero. The intuition is simple. For the first term, it follows because agents take no long positions in trees $j \in J_0^+$, and so in an equilibrium they cannot take short position either. For the second term, it follows because agents take no short positions in trees $j \in J_0^-$, and so the scaling factor θ_j must be equal to zero.

Formally, consider the first term of equation (36). Given that $N_i^+(J_0^+) = 0$ and market clearing, we have $0 = \sum_{i \in I} N_i^+(J_0^+) = \bar{N}(J_0^+) + \sum_{i \in I} N_i^-(J_0^+)$. Hence, we obtain that $N_i^-(J_0^+) = 0$ for all i, so that the first term of (36) is equal to zero.

For the second term of (36), we first use market-clearing and the definition of θ_j to state that for all Borel sets A, $\sum_{i \in I} \int_{j \in A} (1 - \theta_j) dN_{ij}^+ = \sum_{i \in I} \int_{j \in A} \left(dN_{ij}^+ - dN_{ij}^- \right)$. But, since $N_i^-(J_0^-) = 0$, it follows that $N_{ij}^-(A) = 0$ if $A \subseteq J_0^-$, which implies $\int_{j \in A} \theta_j \left(\sum_{i \in I} dN_{ij}^+ \right) \geq 0$ for all Borel sets A, and with an equality if $A \subseteq J_0^-$. The inequality for all Borel sets implies that θ_j is positive almost everywhere according to $\sum_{i \in I} N_i^+$, and hence is positive almost

everywhere according to N_i^+ for each $i \in I$. The equality for $A \subseteq J_0^-$ then implies that θ_j is zero almost everywhere over J_0^- according to N_i^+. It thus follows that the second term of (36) is equal to zero as well, and we are done.

B.2 Proof of Proposition 1

From financial market equilibrium to Arrow Debreu equilibrium. The first part of the Proposition, which states that any financial market equilibrium is the basis of an Arrow-Debreu equilibrium, is an implication of the following Lemma.

Lemma B.1 *A plan (c_i, N_i, a_i) such that $N_i^- = 0$ and $a_i^+(\omega) a_i^-(\omega) = 0$ satisfies the incentive compatibility constraint, (1), the time-zero budget constraint, (2), and the time-one budget constraint, (3), if and only if $a_i^+(\omega) = \max\left\{c_i(\omega) - \int d_j(\omega) dN_{ij}^+, 0\right\}$ and $a_i^-(\omega) = \max\left\{\int d_j(\omega) dN_{ij}^+ - c_i(\omega), 0\right\}$, and*

$$\sum_{\omega \in \Omega} q(\omega) c_i(\omega) + \int p_j \, dN_{ij} \leq \bar{n}_i \int p_j \, d\bar{N}_j + \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega) \, dN_{ij}^+ \tag{37}$$

$$c_i(\omega) \geq \delta \int d_j(\omega) \, dN_{ij}^+ \text{ for all } \omega \in \Omega. \tag{38}$$

For the "only if" part, consider any plan (c_i, a_i, N_i) such that $N_i^- = 0$, $a_i^+(\omega) a_i^-(\omega) = 0$, that satisfies the incentive compatibility constraint (1), the time-zero budget constraint (2) and the time-one budget constraint (3). Taken together, the time-one budget constraint (3) and the assumption that $a_i^+(\omega) a_i^-(\omega) = 0$ imply the stated formulas for $a_i^+(\omega)$ and $a_i^-(\omega)$. The time-one budget constraint also implies that

$$a_i^+(\omega) - a_i^-(\omega) = c_i(\omega) - \int d_j(\omega) \left[dN_{ij}^+ - dN_{ij}^-\right]. \tag{39}$$

Subsituting this expression in the time-zero budget constraint, (3), we obtain the stated inter-temporal budget constraint (37). Lastly, given $N_i^- = 0$, the incentive constraint (1) implies the relaxed constraint $a_i^-(\omega) \leq a_i^+(\omega) + (1-\delta) \int d_j(\omega) \, dN_{ij}^+$. Substituting in this equation the expression for $a_i^+(\omega) - a_i^-(\omega)$ from (39), we obtain (38).

For the "if" part, consider a plan (c_i, N_i, a_i) satisfying (37), (38), $N_i^- = 0$, $a_i^+(\omega) = \max\left\{c_i(\omega) - \int d_j(\omega) dN_{ij}^+, 0\right\}$ and $a_i^-(\omega) = \max\left\{\int d_j(\omega) dN_{ij}^+ - c_i(\omega), 0\right\}$. By construction, the plan satisfies $N_i^- = 0$ and $a_i^+(\omega) a_i^-(\omega) = 0$ and is such that (39) holds. Hence, the time-one budget constraint, (3), holds. Substituting the expression for $c_i(\omega)$ given by the time-one budget constraint, (3), into the intertemporal budget constraint, (37), one then obtain the time-zero budget constraint (2). The only thing left to verify, then, is the incentive constraint (1). If $a_i^-(\omega) = 0$, then the agent has no liability and so the incentive constraint (1) holds trivially. If $a_i^-(\omega) > 0$, then $a_i^+(\omega) = 0$ and $a_i^-(\omega) = \int d_j(\omega) \, dN_{ij}^+ - c_i(\omega)$. Using this expression, we find that the incentive compatibility constraint (1) holds if and only if $\int d_j(\omega) \, dN_{ij}^+ - c_i(\omega) \leq (1-\delta) \int d_j(\omega) \, dN_{ij}^+$, which is the same as (38). The result follows.

From Arrow-Debreu equilibrium to financial market equilibrium. Let us consider an Arrow-Debreu equilibrium $(c_i, N_i^+)_{i \in I}$ and (p, q). Our candidate for a sequential market equilibrium is as follows. The plan for agent $i \in I$ is $\hat{c}_i(\omega) = c_i(\omega)$, $\hat{N}_i^+ = N_i$, $\hat{N}_i^- = 0$,

$$\hat{a}_i^+(\omega) = \max\left\{c_i(\omega) - \int d_j(\omega) \, dN_{ij}^+, 0\right\}$$

$$\hat{a}_i^-(\omega) = \max\left\{\int d_j(\omega) \, dN_{ij}^+ - c_i(\omega), 0\right\}.$$

The price of trees is given by

$$\hat{p}_j = \min\left\{p_j, \sum_{\omega \in \Omega} q(\omega) d_j(\omega)\right\}$$

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and the price of Arrow securities is given by $\hat{q}(\omega) = q(\omega)$. Notice that the price of trees coincides with p_j almost everywhere according to $\bar{N}$.

The candidate allocation clears all markets by construction. The tree market clearing condition implies that agents have zero positions in trees that are in zero supply. Hence, in the Arrow-Debreu equilibrium, the time-zero budget constraint holds for all agents when p is replaced by $\hat{p}$. Since, in addition, $\hat{N}_i^- = 0$ and $\hat{a}_i^+(\omega)\hat{a}_i^-(\omega) = 0$, it follows from an application of Lemma B.1 that $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ satisfies the time-zero and time one budget constraints, (2) and (3) given $(\hat{p}, \hat{q})$. The only thing that remains to be shown is that $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ is optimal given $(\hat{p}, \hat{q})$ in the agent's problem. This is not obvious because, once short-sales are allowed, the budget set in the Arrow-Debreu equilibrium is a subset of the budget set of the financial market equilibrium. That is, we need to show that the plan , we need to show that the plan $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ dominates any other plans, including the one prescribing short sales. To that end we anticipate the result of Proposition C.4 and let λ_i and $\{\mu_i(\omega)\}_{\omega \in \Omega}$ denote the Lagrange multipliers associated with $\hat{c}_i$ and $\hat{N}_i^+$, in the Arrow-Debreu agent's problem, . By construction, we have, for any $(\breve{c}_i, \breve{N}_i, \breve{a}_i)$ satisfying the time zero budget constraint, (2), the time one budget constraint, (3), and the incentive compatibility constraint, (1):

$$\lambda_i \left(\bar{n}_i \int \hat{p}_j \, d\bar{N}_j - \int \hat{p}_j \left[d\breve{N}_{ij}^+ - d\breve{N}_{ij}^- \right] - \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\breve{a}_i^+(\omega) - \breve{a}_i^-(\omega) \right] \right) \geq 0$$

$$\pi(\omega) u_i'(\hat{c}_i(\omega)) \left(\int d_j(\omega) \left[d\breve{N}_{ij}^+ - d\breve{N}_{ij}^- \right] + \breve{a}_i^+(\omega) - \breve{a}_i^-(\omega) - \breve{c}_i(\omega) \right) \geq 0$$

$$\mu_i(\omega) \left((1 - \delta_i(\omega))\breve{a}_i^+(\omega) + \int (1 - \delta_{ij}) d_j(\omega) \, d\breve{N}_{ij}^+ - \breve{a}_i^-(\omega) - \int d_j(\omega) \, d\breve{N}_{ij}^- \right) \geq 0,$$

with equalities if $(\breve{c}_i, \breve{N}_i, \breve{a}_i) = (\hat{c}_i, \hat{N}_i, \hat{a}_i)$. The standard optimality verification argument then implies:

$$
\begin{aligned}
&U_i(\hat{c}_i) - U_i(\breve{c}_i) \\
\geq\ & \sum_{\omega \in \Omega} \pi(\omega) u_i' [\hat{c}_i(\omega)] (\hat{c}_i(\omega) - \breve{c}_i(\omega)) \\
+\ & \lambda_i \left(-\int \hat{p}_j \left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^- \right] - \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) \right] + \int \hat{p}_j \left[d\breve{N}_{ij}^+ - d\breve{N}_{ij}^- \right] + \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\breve{a}_i^+(\omega) - \breve{a}_i^-(\omega) \right] \right) \\
+\ & \sum_{\omega \in \Omega} \pi(\omega) u_i' [\hat{c}_i(\omega)] \left(\int d_j(\omega) \left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^- \right] + \hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) - \hat{c}_i(\omega) - \int d_j(\omega) \left[d\breve{N}_{ij}^+ - d\breve{N}_{ij}^- \right] - \breve{a}_i^+(\omega) + \breve{a}_i^-(\omega) + \breve{c}_i(\omega) \right) \\
+\ & \sum_{\omega \in \Omega} \mu_i(\omega) \Bigg((1 - \delta_i(\omega))\hat{a}_i^+(\omega) + \int (1 - \delta_{ij}) d_j(\omega) \, d\hat{N}_{ij}^+ - \hat{a}_i^-(\omega) - \int d_j(\omega) \, d\hat{N}_{ij}^- \\
& \qquad\qquad - (1 - \delta_i(\omega))\breve{a}_i^+(\omega) - \int (1 - \delta_{ij}) d_j(\omega) \, d\breve{N}_{ij}^+ + \breve{a}_i^-(\omega) + \int d_j(\omega) \, d\breve{N}_{ij}^- \Bigg) \\
=\ & \lambda_i \int (\hat{p}_j - v_{ij}) \left[d\breve{N}_{ij}^+ - d\hat{N}_{ij}^+ \right] + \lambda_i \int \left(\sum_{\omega \in \Omega} q(\omega) - \hat{p}_j \right) \left[d\breve{N}_{ij}^- - d\hat{N}_{ij}^+ \right] + \sum_{\omega} \mu_i(\omega) \delta_i(\omega) \left(\breve{a}_i^+(\omega) - \hat{a}_i^+(\omega) \right),
\end{aligned}
$$

where the equality follows from collecting terms, substituting the expression for v_{ij} given in Proposition C.4, and using the first-order condition $\pi(\omega) u_i' [c_i(\omega)] + \mu_i(\omega) = \lambda_i \hat{q}(\omega)$ in Proposition C.4. Using the first-order conditions with respect to M_{ij} in Proposition C.4, keeping in mind that $\hat{p}_j \leq \sum_{\omega \in \Omega} q(\omega) d_j(\omega)$ by construction, and that $\mu_i(\omega) = 0$ whenever $\hat{a}_i^+(\omega) > 0$, we obtain that the right-hand side above is positive, implying that $U_i(\hat{c}_i) \geq U_i(\breve{c}_i)$. This concludes the proof.

C Proof of the results in Section 4

For the remainder of this appendix we prove all of our results for the generalized model in which δ depends on the agent and (continuously) on the tree type. It is easy to extend the argument of Proposition 1 and show that, when δ is agent- and tree-dependent, then any Arrow-Debreu equilibrium is the basis of a financial market equilibrium, with identical consumption allocation and security prices.

C.1 Proof of Proposition 2

That is, for each, $i \in I$, the function $j \mapsto \delta_{ij}$ is continuous. Let (c, N^+) denote an equilibrium allocation with associated price system (p, q). Suppose it is Pareto dominated by some other incentive-feasible allocation $(\hat{c}, \hat{N}^+)$. Then, because utility is strictly increasing, $\hat{c}_i$ must lie strictly outside the budget set of all agents for which $U_i(\hat{c}_i) > U_i(c_i)$. Otherwise, these agents would have a strict incentive to switch to $\hat{c}_i$. Likewise, $\hat{c}_i$ must lie weakly outside the budget set set of all agents for which $U_i(\hat{c}_i) = U_i(c_i)$. Otherwise, these agents would have strict incentive to increase their consumption in some state, which would respect incentive compatibility. Taken together, we obtain:

$$\sum_{\omega \in \Omega} q(\omega)\hat{c}_i(\omega) + \int p_j \, d\hat{N}_{ij}^+ \geq \bar{n}_i \int p_j \, d\bar{N}_j + \int \sum_{\omega \in \Omega} q(\omega)d_j(\omega) \, d\hat{N}_{ij}^+,$$

with a strict inequality for all $i \in I$ such that $U_i(\hat{c}_i) > U_i(c_i)$. Adding up across all agents we obtain that:

$$\sum_{\omega \in \Omega} q(\omega) \left\{ \sum_{i \in I} \hat{c}_i(\omega) - \sum_{i \in I} \int d_j(\omega) \, d\hat{N}_{ij}^+ \right\} + \int p_j \left\{ \sum_{i \in I} d\hat{N}_{ij}^+ - d\bar{N}_j \right\} > 0,$$

which contradicts the feasibility of $(\hat{c}, \hat{N})$.

C.2 Proof of Proposition 3

Our proof of existence proceeds as follows. In Section C.2.1 we define the Planner's Problem, we study some of its elementary properties, and we derive necessary and sufficient optimality conditions for a solution. In Section C.2.2, we turn to the equilibrium and derive first-order necessary and sufficient conditions for a solution to the agent's problem. Comparing the first-order conditions for the Planner and for the agent, in Section C.2.3 we show an equivalence between the set of equilibrium allocations, and the set of solutions to the Planner's problem with zero wealth transfers. We then establish the existence of a solution to the Planner's problem with zero wealth transfer. Omitted proofs are in Supplementary Appendix E.

In what follows we identify any measure with its cumulative distribution function. That is, we identify $\mathcal{M}_+$ with the set of increasing and right-continuous functions over $[0, 1]$. We denote by $\mathcal{M}$ the vector space of functions which can be written as $F = F_1 - F_2$, where both F_1 and F_2 belong to $\mathcal{M}_+$. We endow $\mathcal{M}$ with the total variation norm. Given any sequence $N^k \in \mathcal{M}$, we said that N^k *converges strongly* towards N, and write $N^k \to N$, if $\lim_{k \to \infty} \|N^k - N\| = 0$. We say that N^k *converges weakly* towards N, and write $N^k \Rightarrow N$, if $\int f_j \, dN_j^k \to \int f_j \, dN_j$ for all continuous real-valued functions $j \mapsto f_j$ over $[0, 1]$. A set of allocations K is said to be *weakly closed* if for any weakly converging sequence $(c^k, N^k) \in K$, i.e. such that $c^k \to c$ and $N^k \Rightarrow N$, then the limit of the sequence belongs to K, i.e., $(c, N) \in K$. The set K is said to be *weakly compact* if for any sequence $(c^k, N^k) \in K$, there exist some subsequence (c^ℓ, N^ℓ) and some $(c, N) \in K$ such that $c^\ell \to c$ and $N^\ell \Rightarrow N$.

C.2.1 The Planner's Problem

Let $\mathcal{A}$ denote the simplex, i.e., the set of welfare weights $\alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_I)$ such that $\alpha_i \geq 0$ and $\sum_{i \in I} \alpha_i = 1$. Given any $\alpha \in \mathcal{A}$, and given any allocation (c, N^+), social welfare is defined as

$$W(\alpha, c, N) \equiv \sum_{i \in I} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i(\omega) \right].$$

In the above formula, when $u_i(0) = -\infty$, we let $\alpha_i u_i \left[c_i(\omega) \right] = 0$ if $\alpha_i = c_i(\omega) = 0$.

Given weight $\alpha \in \mathcal{A}$, the *Planner's Problem* is:

$$W^\star(\alpha) = \sup W(\alpha, c, N^+) \tag{40}$$

with respect to incentive-feasible allocations, i.e., with respect to all allocations (c, N^+) satisfying (38), (9) and (8). We let $\Gamma^\star(\alpha)$ denote the set of allocations solving (40). To show the existence of a solution, we rely on:

Lemma C.1 *The set of incentive feasible allocations is weakly compact.*

The proof relies on Helly's Selection Theorem (Theorem 12.9 in Stokey and Lucas (1989)) which allows to extract weakly convergence subsequences from bounded sequences in $\mathcal{M}_+$. The feasibility and incentive compatibility constraints hold in the limit by definition of weak convergence. We add to the argument in Stokey and Lucas (1989) by showing that the feasibility constraint for tree holdings is also satisfied in the limit. With this result in mind, we show in the supplementary appendix:

Proposition C.1 *The planner's value $W^\star(\alpha)$ is a continuous function of $\alpha \in \mathcal{A}$, and the maximum correspondence $\Gamma^\star(\alpha)$ is non-empty, weakly compact, convex, and has a weakly closed graph. Moreover, consider any sequence $\alpha^k \to \bar{\alpha}$ and an associated sequence of optimal allocations $(c^k, N^{k+}) \in \Gamma^\star(\alpha^k)$. Then, if $\bar{\alpha}_i = 0$, $\lim_{k \to \infty} \alpha_i^k u' \left[c_i^k(\omega) \right] c_i^k(\omega) = 0$ for all $\omega \in \Omega$.*

If $u_i(0) = 0$ for all $i \in I$, the result follow from the same argument as in the proof of the Theorem of the Maximum (see, for example, Theorem 3.6 in Stokey and Lucas (1989)). If $u_i(0) = -\infty$ for some i, then we need to adapt the argument because the social welfare function is not continuous at (α, c, N) such that $\alpha_i = c_i(\omega) = 0$. Likewise, the result concerning $\alpha_i^k u' \left[c_i^k(\omega) \right] c_i^k(\omega) = 0$ is obvious if $u_i(0) = 0$, but requires some additional work when $u_i(0) = -\infty$.

To compare equilibria with solutions of the Planner's Problem, we rely on first-order conditions. We first derive necessary conditions. To do so, we cannot apply the Lagrange multiplier theorems of Luenberger (1969), because they do not accommodate equality constraints. Even if we consider a "relaxed problem" where equality constraints are replaced by inequality constraints, the theorems do not apply because the relevant positive cone has an empty interior. We therefore exploit the structure of the problem to derive first-order conditions by hand. To do so we consider, for any N, the maximized objective with respect to c. We then use an Envelope Theorem of Milgrom and Segal (2002) to explicitly calculate the directional derivative of this maximized objective with respect to N. We obtain:

Proposition C.2 *Suppose (c, N^+) solves the Planner's problem given some $\alpha \in \mathcal{A}$. Then there exists multipliers $\hat{q} \in \mathbb{R}_+^{|\Omega|}$ and $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$ such that (c, N^+) satisfies two sets of conditions.*

- *First-order conditions:*

$$\alpha_i \pi(\omega) u_i' \left[c_i(\omega) \right] + \hat{\mu}_i(\omega) = \hat{q}(\omega), \quad \forall (i, \omega) \in I \times \Omega$$

$$\int \left[\hat{p}_j - \hat{v}_{ij} \right] dN_{ij}^+ = 0,$$

where $\hat{v}_{ij} \equiv \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega)$, and $\hat{p}_j \equiv \max_{i \in I} \hat{v}_{ij}$.

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- *Complementary slackness conditions:*

$$\hat{q}(\omega) \left[\sum_{i \in I} \int d_j(\omega) \, dN_{ij}^+ - \sum_{i \in I} c_i(\omega) \right] = 0 \quad \forall \omega \in \Omega$$

$$\hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) \, dN_{ij}^+ \right] = 0 \quad \forall (i, \omega) \in I \times \Omega.$$

Although the above conditions are also sufficient, it is convenient to state more general sufficient conditions, where $\hat{p}$ is taken to be some abstract continuous linear functional. This allows to show that any equilibrium is a solution to the Planner's Problem, even if the pricing functional cannot be represented by a continuous function. Then, using the necessary conditions derived in Proposition C.2, one can show that the same equilibrium allocation can be supported by a pricing functional represented by a continuous function.

Proposition C.3 *An incentive-feasible allocation (c, N^+) solves the Planner's problem if there exist multipliers $\hat{q} \in \mathbb{R}_+^{|\Omega|}$, $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$, and a continuous linear functional $\hat{p}$ satisfying the following two sets of conditions.*

- *First-order conditions:*

$$\alpha_i \pi(\omega) u_i' [c_i(\omega)] + \hat{\mu}_i(\omega) = \hat{q}(\omega), \quad \forall (i, \omega) \in I \times \Omega$$

$$\hat{p} \cdot M^+ - \int \hat{v}_{ij} \, dM_{ij}^+ \geq 0 \quad \forall M_i^+ \in \mathcal{M}_+ \text{ and } i \in I, \text{ with } "=" \text{ if } M^+ = N_i^+,$$

where $\hat{v}_{ij} \equiv \sum_{\omega \in \Omega} q(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \mu_i(\omega) \delta_{ij} d_j(\omega)$.

- *Complementary slackness conditions:*

$$\hat{q}(\omega) \left[\sum_{i \in I} \int d_j(\omega) \, dN_{ij}^+ - \sum_{i \in I} c_i(\omega) \right] = 0 \quad \forall \omega \in \Omega$$

$$\hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) \, dN_{ij}^+ \right] = 0 \quad \forall (i, \omega) \in I \times \Omega.$$

C.2.2 Optimality conditions for the Agent's Problem

Notice that the range of the constraint set in the agent's problem is finitely dimensional. In this case, the "interior point condition" for the positive cone associated with the constraint set is immediately satisfied and so one can apply the general Lagrange multiplier theorems shown in Section 8.3 and 8.4 of Luenberger (1969).

Proposition C.4 *A plan (c_i, N_i^+) solve the agent's problem if and only if it satisfies the intertemporal budget constraint, (37), the incentive compatibility constraint (38), and there exists multipliers $\lambda_i \in \mathbb{R}_+$, $\mu_i \in \mathbb{R}_+^{|\Omega|}$ satisfying the following two sets of conditions:*

- *First-order conditions:*

$$\pi(\omega) u_i' [c_i(\omega)] + \mu_i(\omega) = \lambda_i q(\omega)$$

$$\int (p_j - v_{ij}) \, dM_{ij}^+ \geq 0 \quad \forall M_i^+ \in \mathcal{M}_+, \text{ with } "=" \text{ if } M_i^+ = N_i^+,$$

where $v_{ij} \equiv \sum_{\omega \in \Omega} q(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \frac{\mu_i(\omega)}{\lambda_i} \delta_{ij} d_j(\omega)$.

- *Complementary slackness conditions:*

$$\lambda_i \left[\bar{n}_i \int p_j \, d\bar{N}_j + \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega) \, dN_{ij}^+ - \int p_j \, dN_{ij}^+ - \sum_{\omega \in \Omega} q(\omega) c_i(\omega) \right] = 0$$

$$\mu_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) \, dN_{ij}^+ \right] = 0 \quad \forall \omega \in \Omega.$$

There is one difference between this Proposition and the Theorems shown in Section 8.3 and 8.4 of Luenberger (1969): we are asserting that there exists multipliers such that the first-order condition with respect to $c_i(\omega)$ holds with equality. This follows from the following observation: if $c_i(\omega) = 0$, then the incentive compatibility constraint is binding, in particular $\int \delta_{ij} d_j(\omega) \, dN_{ij}^+ = 0$. Therefore, if we raise $\mu_i(\omega)$ so that the first-order condition holds with equality, we leave the product $\mu_i(\omega) \int \delta_{ij} d_j(\omega) \, dN_{ij}^+ = 0$ unchanged, which implies that $\int (p_j - v_{ij}) \, dN_{ij} = 0$ continues to hold. Finally, since raising $\mu_i(\omega)$ decreases v_{ij}, $\int (p_j - v_{ij}) \, dM_{ij}^+$ remains positive. Taken together, this means that we can always pick multipliers so that the first-order condition with respect to $c_i(\omega)$ holds with equality.

Finally, the following result provide a simple relationship between the value of the agent's endowment, and the marginal value of his consumption plan. This formula will be useful shortly to formulate the equilibrium fixed-point equation.

Lemma C.2 *If (c_i, N_i^+) solves the agent's problem, then*

$$\sum_{\omega \in \Omega} \pi(\omega) u' \left[c_i(\omega) \right] c_i(\omega) = \lambda_i \bar{n}_i \int p_j \, d\bar{N}_j.$$

The proof of the Lemma goes as follows. A solution to the agent's problem, (c_i, N_i^+), maximizes the Lagrangian:

$$
\begin{aligned}
L(\hat{c}_i, \hat{N}_i^+) = U_i(\hat{c}_i) \quad &+ \quad \lambda_i \left[\bar{n}_i \int p_j \, d\bar{N}_j + \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega) \, d\hat{N}_{ij} - \int p_j \, d\hat{N}_{ij}^+ - \sum_{\omega \in \Omega} q(\omega) c_i(\omega) \right] \\
&+ \quad \sum_{\omega \in \Omega} \mu_i(\omega) \left[\hat{c}_i(\omega) - \int \delta_{ij} d_j(\omega) \, d\hat{N}_{ij} \right],
\end{aligned}
$$

with respect to $(\hat{c}_i, \hat{N}_i)$. This implies that the function $\beta \mapsto L(\beta c_i, \beta N_i)$ is maximized at $\beta = 1$. Taking first-order condition with respect to β at $\beta = 1$, and using the complementary slackness conditions, yields the desired result.

C.2.3 Existence of a Planner's Solution with Zero Wealth Transfer

By comparing the first-order conditions of the Planner and of the agent, we obtain:

Proposition C.5 *An allocation (c, N^+) is an Arrow-Debreu equilibrium allocation if and only if there exists $\alpha \in \mathcal{A}$ such that:*

- *(c, N^+) solves the Planner's problem given α;*
- *For all $i \in I$, $\alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i' \left[c_i(\omega) \right] c_i(\omega) = \bar{n}_i \sum_{k \in I} \sum_{\omega \in \Omega} \pi(\omega) u_k' \left[c_k(\omega) \right] c_k(\omega).$*

In particular, given a solution of the Planner's problem satisfying the above two conditions, an equilibrium price system is given by the multipliers $(\hat{p}, \hat{q})$ of Proposition C.2.

Intuitively, comparing the first-order conditions of the Planner and of the agent reveals that the weight α_i must be proportional to $1/\lambda_i$, the inverse of the Lagrange multiplier on the agent's budget constraint. It then follows from

Lemma C.2 that, for all agents $i \in I$:

$$\alpha_i \sum_{\omega \in \Omega} \pi(\omega) u'_i [c_i(\omega)] = \bar{n}_i \times \left[\sum_{k \in I} \frac{1}{\lambda_k} \right]^{-1} \times \int p_j \, d\bar{N}_j.$$

The second condition then follows because $\sum_{i \in I} \bar{n}_i = 1$. The final result about the price system follows from direct comparison of the first-order conditions of the agent and the planner.

We are now ready to establish the existence of an equilibrium. Let $\Delta^\star(\alpha)$ denote the set of transfers $\{\Delta_i^\star(\alpha)\}_{i \in I}$ such that:

$$\Delta_i^\star(\alpha) \equiv \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u'_i [c_i(\omega)] c_i(\omega) - \bar{n}_i \sum_{k \in I} \alpha_k \sum_{\omega \in \Omega} \pi(\omega) u'_k [c_k(\omega)] c_k(\omega), \tag{41}$$

generated by all $(c, N^+) \in \Gamma^\star(\alpha)$, with the convention that $\alpha_i u'_i(c)c = 0$ if $\alpha_i = c = 0$. Using the Kakutani's fixed-point Theorem, as in Negishi (1960) and Magill (1981), we can show:

Proposition C.6 *There exists some $\alpha \in \mathcal{A}$, such that $0 \in \Delta^\star(\alpha)$.*

Based on some $\alpha \in \mathcal{A}$, using Proposition C.5, we can construct an equilibrium allocation and price system.

C.3   First Best Implementability

The objective of this appendix is to study circumstances under which the incentive compatibility constraints do not impact equilibrium outcomes. Formally, define a $\delta = 0$ equilibrium to be a first-best allocation and price system (c^0, N^{0+}, p^0, q^0) when $\delta = 0$, i.e., when there is no scope for incentive problems. Fix some $\delta > 0$. Then, the $\delta = 0$-equilibrium is said to be $\delta > 0$-*implementable* if there exists some $\delta > 0$-equilibrium, $(c^\delta, N^{\delta+}, q^\delta, p^\delta)$, such that $c^0 = c^\delta$. It is clear that a $\delta = 0$-equilibrium, (c^0, N^{0+}, p^0, q^0), is $\delta > 0$-implementable if and only if there exists a feasible tree allocation, $N^{\delta+}$, such that incentive constraints are satisfied for all agents and all states:

$$\sum_{i \in I} N_i^{\delta+} = \bar{N} \tag{42}$$

$$c_i^0(\omega) \geq \delta \int d_j(\omega) \, dN_{ij}^{\delta+} \quad \forall (i, \omega) \in I \times \Omega. \tag{43}$$

The feasibility condition (42) is crucial: in its absence, the incentive constraint would not have any bite, since it would be trivial to satisify (43) by setting $N_i^+ = 0$ for all agents. This observation means that, in our model, binding incentive compatibility constraints is ultimately a general equilibrium phenomenon. This is in contrast with earlier models in which non-pledgeable income is not tradeable: in these environments, binding incentive constraints would already arise in partial equilibrium contract-theoretic settings.

Using conditions (42) and (43), we obtain:

Proposition C.7 *Fix some $\delta > 0$. A $\delta = 0$ equilibrium (c^0, N^{0+}, p^0, q^0) is $\delta > 0$-implementable if one of the following conditions is satisfied:*

- *Inada conditions are satisfied for all $i \in I$ and δ is strictly positive but small enough.*
- *There exists $\{N_i^+\}_{i \in I} \in \mathcal{M}_+^{|I|}$ such that $\sum_{i \in I} N_i^+ = \bar{N}$ and $\int d_j(\omega) \, dN_{ij}^+ = c_i^0(\omega) \, \forall (i, \omega) \in I \times \Omega$.*
- *Agents have Constant Relative Risk Aversion (CRRA) with identical coefficient.*

To understand the first bullet point, note that, with Inada conditions, consumptions are strictly positive for all agents and all states. Therefore, as long as δ is small enough, the incentive compatibility constraint (43) is satisfied for all agents and all states when each agent holds, say, an equal fraction of the market portfolio, $N_i^+ = \bar{N}/|I|$, and simultaneously issues liabilities to attain its first-best consumption plan.

The second bullet point of the proposition states that all incentive compatibility constraints hold if two sets of conditions are satisfied. First agents can replicate their $\delta = 0$-equilibrium consumption with *positive* holdings of trees. Second, these agents' holdings are *feasible*, i.e., they add up to the aggregate. This means that they do not need to make any financial promise, i.e., promise to deliver consumption out of the payoff of their equilibrium holdings of trees. Clearly, if agents do not need to make any financial promise, limited pledgeability is not an issue.

The third bullet point is an example of the second: if agents have CRRA utilities with identical risk aversion, then they all consume a constant share of the aggregate endowment. Clearly, they can attain that consumption plan by holding a portfolio of trees, namely a constant share in the market portfolio.

Equations (42)-(43) and Proposition C.7 also help understand circumstances under which a $\delta = 0$ equilibrium *cannot* be implemented. Consider for example an economy composed of CRRA utility agents with heterogenous risk aversion, and assume that there is only one tree, the "market portfolio", with payoff equal to aggregate consumption. Because of heterogeneity in risk aversion, in the $\delta = 0$ equilibrium, agents' consumption shares vary across states – namely more risk averse agents tend to have higher consumption shares in states of low aggregate consumption. If $\delta \simeq 1$, agents cannot issue liabilities and their consumption must be approximately equal to the payoff of their tree portfolio. But since they can only hold the market portfolio, their consumption share must be approximately constant across states, so that the $\delta \simeq 1$ equilibrium cannot coincide with the $\delta = 0$ equilibrium.

C.4 Uniqueness in the CRRA ≤ 1 case

Proposition C.8 *Suppose that there are two types of agents, $I = \{1, 2\}$, with CRRA utility, with respective RRA coefficients (γ_1, γ_2) such that $0 \leq \gamma_1 \leq \gamma_2 \leq 1$ and $\gamma_2 > 0$. Then the Arrow-Debreu equilibrium consumption allocation is uniquely determined. The prices of consumption claims, q, the price of trees, p, are uniquely determined $\bar{N}$-almost everywhere up to a positive multiplicative constant.*

In general, the tree allocation is not uniquely determined. As will be clear below, this arises for example when none of the incentive constraints bind. In that case the allocation is not uniquely determined because it is equivalent to hold tree j or a portfolio of Arrow securities with the same cash-flows as j.

As is standard, only relative prices are pinned down, hence price levels are only determined up to a positive multiplicative constant.

In an Arrow-Debreu equilibrium, tree prices are only uniquely determined $\bar{N}$-almost everywhere. In particular, the prices of trees in zero net supply are not uniquely determined. This is intuitive: given the short-sale constraint, the only equilibrium requirement for a tree in zero supply is that the price is large enough so that no agent want to hold it. As a result equilibrium only imposes a lower bound on the price of trees in zero supply.[24]

Step 1: an equivalent one-equation-in-one-unknown problem. Since the utility function of agent $i = 2$ is strictly concave, its allocation is uniquely determined in the Planner's problem. But since $c_1(\omega) + c_2(\omega) = \int d_j(\omega) \, d\bar{N}_j$, the consumption allocation of agent 1 is also uniquely determined. Hence $\Delta^\star(\alpha)$, defined in equation (41), is a function and not a correspondence. Moreover since $\Delta_1^\star(\alpha) + \Delta_2^\star(\alpha) = 0$ by construction and $\alpha_1 + \alpha_2 = 1$ by assumption, it is enough to look for a solution of $\Delta_1^\star(\alpha_1, 1 - \alpha_1) = 0$. That is, solving for equilibrium boils down to a one-equation in one-unknown problem. To formulate this problem in simple terms, let

$$\mathrm{MU}_i(c_i) \equiv \sum_\omega \pi(\omega) u_i' \left[c_i(\omega) \right] c_i(\omega).$$

---

[24]To remove the indeterminacy, it would be natural to inject a small additional supply for all trees, $\bar{N} + \varepsilon \mathcal{U}_{[0,1]}$ and let $\varepsilon \to 0$. In addition, as mentioned earlier, the implementation of an Arrow equilibrium as a security market equilibrium rules out very high prices, and so reduces the indeterminacy.

Notice, that with CRRA utility, $\text{MU}_i(c_i) = (1 - \gamma_i)U_i(c_i)$ for $\gamma_i \neq 1$, and $\text{MU}_i(c_i) = 1$ for $\gamma_i = 1$. With this notation, the one-equation-in-one-unknown problem for equilibrium is:

$$\bar{n}_2 \alpha_1 \text{MU}_1(c_1) - \bar{n}_1 \alpha_2 \text{MU}_2(c_2) = 0, \tag{44}$$

where (c_1, c_2) is the consumption allocation chosen by the planner given weight $\alpha \in \mathcal{A}$. We already know from Proposition C.6 that this equation has a solution.

Step 2: an intermediate result. An intermediate result for proof of uniqueness is the observation that, for any α' and α such that $\alpha_1' > \alpha_1$, and for all for all $c \in \Gamma^\star(\alpha)$ and $c' \in \Gamma^\star(\alpha')$.

$$U_1(c_1') \geq U_1(c_1) \text{ and } U_2(c_2') \leq U_2(c_2)$$
$$\text{MU}_1(c_1') \geq \text{MU}_1(c_1) \text{ and } \text{MU}_2(c_2') \leq \text{MU}_2(c_2)$$

To prove this intermediate result, consider two sets of weights α and α' with corresponding optimal allocations $(c, N^+) \in \Gamma^\star(\alpha)$ and $(c', N'^+) \in \Gamma^\star(\alpha')$. Since the constraint set of the planner does not depend on α, (c, N^+) and (c', N'^+) are both incentive feasible given α and α'. Hence, optimality implies that:

$$\alpha_1 U_1(c_1) + \alpha_2 U_2(c_2) \geq \alpha_1 U_1(c_1') + \alpha_2 U_2(c_2') \Leftrightarrow \alpha_1 \left[U_1(c_1) - U_1(c_1') \right] + \alpha_2 \left[U_2(c_2) - U_2(c_2') \right] \geq 0.$$

Vice versa:
$$\alpha_1' \left[U_1(c_1') - U_1(c_1) \right] + \alpha_2' \left[U_2(c_2') - U_2(c_2) \right] \geq 0.$$

Adding up these two inequality and using that, since the weight add up to one, $\alpha_1' - \alpha_1 = \alpha_2 - \alpha_2'$, we obtain:

$$\left[\alpha_1' - \alpha_1 \right] \left\{ \left[U_1(c_1') - U_1(c_1) \right] - \left[U_2(c_2') - U_2(c_2) \right] \right\},$$

which implies that:
$$U_1(c_1') - U_1(c_1) \geq U_2(c_2') - U_2(c_2).$$

But then we must have that
$$U_1(c_1') - U_1(c_1) \geq 0 \geq U_2(c_2') - U_2(c_2).$$

because otherwise either (c, N) or (c', N') would not be constrained Pareto optima.

Step 3: equation (44) has a unique solution. We now go back to equation (44). Let α denote some solution, and consider any $\alpha' \neq \alpha$, for example such that $\alpha_1' > \alpha_1$. Let c and c' denote the consumption allocations associated with α and α'. Then,

$$\begin{aligned}
& \bar{n}_2 \alpha_1' \text{MU}_1(c_1') - \bar{n}_1 \alpha_2' \text{MU}_2(c_2') \\
= \; & \bar{n}_2 \alpha_1' \text{MU}_1(c_1') - \bar{n}_1 \alpha_2' \text{MU}_2(c_2') - \bar{n}_2 \alpha_1 \text{MU}_1(c_1) + \bar{n}_1 \alpha_2 \text{MU}_2(c_2) \\
= \; & \bar{n}_2 \alpha_1' \left[\text{MU}_1(c_1') - \text{MU}_1(c_1) \right] - \bar{n}_1 \alpha_2' \left[\text{MU}_2(c_2') - \text{MU}_2(c_2) \right] + \left(\alpha_1' - \alpha_1 \right) \left[\bar{n}_2 \text{MU}_1(c_1) + \bar{n}_1 \text{MU}_2(c_2) \right] > 0.
\end{aligned}$$

In the above, the second line follows from subtracting $\bar{n}_2 \alpha_1 \text{MU}_1(c_1) - \bar{n}_1 \alpha_2 \text{MU}_2(c_2) = 0$ since α was assumed to solve (44). The third line follows from re-arranging terms and keeping in mind that $\alpha_1' - \alpha_1 = \alpha_2 - \alpha_2'$. The inequality follows from the intermediate result established in Step 2, and from the fact that marginal utilities are strictly positive. Vice versa, if we consider some $\alpha' \neq \alpha$ such that $\alpha_1' < \alpha_1$, we obtain that the equilibrium equation

(44) is strictly negative. Therefore, the equation for the weight, α, has a unique solution.

Step 4: the various uniqueness claims. Consider any equilibrium allocation, (c, N^+), and price system, (p, q). From Proposition C.5, we know that (c, N^+) solves the Planner's given the unique set of weights such that $\Delta^\star(\alpha) = 0$. But, as argued above, the consumption allocation is uniquely determined in the Planner's problem. Hence, it follows that the equilibrium allocation is uniquely determined in an equilibrium as well. Next, by direct comparison of first-order conditions, one sees that (c, N) solve the first-order conditions of the Planner's problem with weights $\alpha_i = \beta/\lambda_i$, multipliers $\hat{q}(\omega) = \beta q(\omega)$, $\hat{\mu}_i(\omega) = \alpha_i \mu_i(\omega)$, $\hat{v}_{ij} = \beta v_{ij}$ and $\hat{p}_j = \beta p_j$, where λ_i is the Lagrange multiplier on agent's i budget constraint, and $\beta \equiv \left[\sum_{k \in I} 1/\lambda_k \right]^{-1}$. But from the first-order conditions of the Planner's problem, and given that c is uniquely determined, it follows that $\hat{q}(\omega)$, $\hat{\mu}(\omega)$ and $\hat{v}_{ij}$ are uniquely determined as well. Clearly, this implies that the price of Arrow securities, q, and the private tree valuations, v, are uniquely determined up to the multiplicative constant $1/\beta$. Now turning to the price of trees, we note that the first-order condition of the agent's problem imply that $p_j = v_{ij}$ for almost all trees held by i. Since the private valuations are uniquely determined up to the multiplicative constant $1/\beta$, the same property must hold for the price trees $\bar{N}$-almost everywhere.

C.5 Proof of Proposition 4

Suppose the equilibrium is not first best. Consider an agent i. There must exist some state in which agent i's incentive constraint binds, otherwise it would be possible to reallocate a small fraction of all tree supplies to agent i while still satisfying his incentive constraint, and that would make all other agents' incentive constraints slack, contradicting that the equilibrium is not first best.

Consider state ω in which agent i's incentive constraint binds. There must exist some other agent i' whose incentive constraint is slack in state ω. Indeed, adding up the incentive constraint (6) across all agents and using market clearing for consumption, one immediately sees that the aggregate incentive constraint must be slack in each state. Consider $\varepsilon > 0$ and tree j such that $d_j(\omega')/d_j(\omega) < \varepsilon$ for all $\omega' \neq \omega$. Then:

$$
\begin{aligned}
p_j &\geq \sum_{\omega' \in \Omega} q(\omega') d_j(\omega') - \sum_{\omega' \in \Omega} \frac{\mu_{i'}(\omega')}{\lambda_{i'}} \delta d_j(\omega') \\
&= \sum_{\omega' \in \Omega} q(\omega') d_j(\omega') - \sum_{\omega' \neq \omega} \frac{\mu_{i'}(\omega')}{\lambda_{i'}} \delta d_j(\omega') \\
&\geq \sum_{\omega' \in \Omega} q(\omega') d_j(\omega') - \sum_{\omega' \neq \omega} \frac{\mu_{i'}(\omega')}{\lambda_{i'}} \varepsilon \delta d_j(\omega) \\
&> \sum_{\omega' \in \Omega} q(\omega') d_j(\omega') - \frac{\mu_i(\omega)}{\lambda_i} \delta d_j(\omega) \\
&\geq \sum_{\omega' \in \Omega} q(\omega') d_j(\omega') - \sum_{\omega' \in \Omega} \frac{\mu_i(\omega')}{\lambda_i} \delta d_j(\omega') \\
&= v_{ij},
\end{aligned}
$$

where the first line is equation (12) for agent i, the second line obtains because agent i's incentive constraint is slack in state ω, the third line obtains because $d_j(\omega')/d_j(\omega) < \varepsilon$ for all $\omega' \neq \omega$, the fourth line obtains by picking some ε such that $\varepsilon < \frac{\mu_i(\omega)}{\lambda_i} / \sum_{\omega' \neq \omega} \frac{\mu_{i_0}(\omega')}{\lambda_i}$, the fifth line obtains because $\mu_i(\omega') \geq 0$ for all ω', and the sixth line is equation (12) for agent i.

C.6 Proof of Proposition 5

Consider agent i holding tree j. His valuation for the tree must equal the tree price:

$$
\begin{aligned}
p_j &= \sum_{\omega \in \Omega} q(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \frac{\mu_i(\omega)}{\lambda_i} \delta d_j(\omega) \\
&= \sum_{\omega \in \Omega} \left[q(\omega) - \frac{\mu_i(\omega)}{\lambda_i} \delta \right] \left[\int d_k(\omega) dX_k + Y(\omega) \right] \\
&= \int v_{ik} dX_k + \sum_{\omega \in \Omega} \left[q(\omega) - \frac{\mu_i(\omega)}{\lambda_i} \delta \right] Y(\omega) \\
&< \int p_k dX_k + \sum_{\omega \in \Omega} q(\omega) Y(\omega),
\end{aligned}
$$

where we move to the second line by substituting the tree payoff with the payoff of the replicating portfolio, we move to the third line using expression (12) for agent i's valuation for trees k, and we move to the fourth line using that agent i is not willing to hold all assets in the replicating portfolio, which implies $\int v_{ik} dX_k < \int p_k dX_k$ or $\mu_i(\omega) Y(\omega) > 0$ for some ω.

D Proof of the results in Section 5

D.1 Proof of Lemma 2

As before we state proofs for our results when δ_{ij} is assumed to depend both on the type of agent holding the tree and on the type of the tree. In this case, the Proposition holds under the additional restriction that:

$$
\frac{\delta_{1j} d_j(\omega_1)}{\delta_{2j} d_j(\omega_2)}, \tag{45}
$$

is strictly increasing. Notice that this restriction is automatically satisfied whenever $\delta_{1j} = \delta_{2j}$ for all j. The generalization of (20)-(21) is

$$
c_1(\omega_1) \geq \int_{j \in [0,k)} \delta_{1j} d_j(\omega_1) d\bar{N}_j + \delta_{1k} d_k(\omega_1) \Delta N_1 \tag{46}
$$

$$
c_2(\omega_2) \geq \int_{j \in (k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j + \delta_{2k} d_k(\omega_2) \Delta N_2. \tag{47}
$$

The "if" part of the Proposition. Pick the smallest possible k and the largest possible ΔN_2 such that the inequalities (46)-(47) hold. Consider the corresponding tree allocation $N_1 = \bar{N} \mathbb{I}_{\{j \in [0,k)\}} + \Delta N_1 \mathbb{I}_{\{j=k\}}$ and $N_2 = \Delta N_2 \mathbb{I}_{\{j=k\}} + \bar{N} \mathbb{I}_{\{j \in (k,1]\}}$.

By construction, the incentive constraint of agent $i = 1$ holds in state ω_1, and the incentive constraint of agent $i = 2$ holds in state ω_2.

Next, we argue that the incentive constraint of agent $i = 1$ holds in state ω_2. This is obvious if N allocates no tree to agent $i = 1$. Otherwise, if N allocates some trees to agent $i = 1$, then the incentive constraint of agent $i = 2$ must bind in state ω_2. Given $\delta_{ij} < 1$, this implies that the incentive constraint of agent $i = 1$ holds in state ω_2.

With the above observations in mind, the only incentive constraint that remains to be checked is that of agent

$i = 2$ in state ω_1. If it holds with the proposed tree allocation, N, we are done. Otherwise,

$$c_2(\omega_1) < \int_{(k,1]} \delta_{2j} d_j(\omega_1)\, d\bar{N}_j + \delta_{2k} d_{2k} \Delta N_2,$$

in which case we explicitly construct another allocation of tree holdings that is incentive compatible. We proceed as follows. We start from the proportional tree allocation that delivers agents $i = 1$ and $i = 2$ their consumption in state ω_2: $\hat{N}_1 = \frac{c_1(\omega_2)}{y(\omega_2)} \bar{N}$ and $\hat{N}_2 = \frac{c_2(\omega_2)}{y(\omega_2)} \bar{N}$. By construction, with such proportional allocation, the incentive constraint of both agents hold in state ω_2. Since the consumption share of agent $i = 2$ is strictly larger in state ω_1 than in state ω_2, one sees that that agent $i = 2$ incentive compatibility constraint is slack in state ω_1. Indeed, we have:

$$c_2(\omega_1) > \frac{y(\omega_1)}{y(\omega_2)} c_2(\omega_2) = \frac{c_2(\omega_2)}{y(\omega_2)} \int d_j(\omega_1) d\bar{N}_j = \int d_j(\omega_1) d\tilde{N}_{2j} > \int \delta_{2j} d_j(\omega_1) d\tilde{N}_{2j},$$

where the first inequality states that the consumption share is larger in state ω_1 than in state ω_2, the first equality follows from rearranging and from the definition of $y(\omega_1)$, the second equality follows from the definition of N_2, and the last inequality follows because $\delta_{2j} < 1$.

Taking stock, for the original allocation N, the incentive compatibility constraints hold in state ω_2 for both $i = 1$ and $i = 2$, but it does not hold in state ω_1 for agent $i = 2$. For the proportional allocation $\hat{N}$, the incentive compatibility constraints also hold in state ω_2 for both $i = 1$ and $i = 2$, and it holds with strict inequality in state ω_1 for agent $i = 2$. Therefore, there is a convex combination of N and $\hat{N}$ such that the incentive compatibility constraint is binding in state ω_1 for agent $i = 2$. This implies that the incentive compatibility constraint holds in state ω_1 for agent $i = 1$. Clearly, the incentive compatibility constraint also hold in state ω_2 for both agents since they hold separately for N and $\hat{N}$.

The "only if" part of the Proposition. As before, pick the smallest possible k and the largest possible ΔN_2 such that (47) holds. Given this ΔN_2, let $\Delta N_1 \equiv \hat{N}_k - \bar{N}_{k-} - \Delta N_2$. If $k = 0$ and $\Delta N_2 = \bar{N}_0$, then (46) evidently holds. Otherwise, (47) holds with equality and we need to establish that that (46) holds as well. To that end, consider any $\hat{N}$ such that $(c, \hat{N})$ is incentive feasible. Then:

$$
\begin{aligned}
&\int_{[0,k)} \delta_{1j} d_j(\omega_1) d\bar{N}_j + \delta_{1k} d_k(\omega_1) \Delta N_1 \\
=~ &\int_{[0,k)} \delta_{1j} d_j(\omega_1) \left(d\hat{N}_{1j} + d\hat{N}_{2k} \right) + \delta_{1k} d_k(\omega_1) \Delta N_1 \\
=~ &\int_{[0,1]} \delta_{1j} d_j(\omega_1) d\hat{N}_{1j} - \int_{[k,1]} \delta_{1j} d_j(\omega_1) d\hat{N}_{1j} + \delta_{1k} d_k(\omega_1) \Delta N_1 + \int_{[0,k)} \delta_{1j} d_j(\omega_1) d\hat{N}_{2j} \\
\leq~ &c_1(\omega_1) - \int_{[k,1]} \delta_{2j} d_j(\omega_2) \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} d\hat{N}_{1j} + \delta_{1k} d_k(\omega_1) \Delta N_1 + \int_{[0,k)} \delta_{2j} d_j(\omega_2) \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} d\hat{N}_{2j} \\
=~ &c_1(\omega_1) + \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} \left[\int_{[0,1]} \delta_{2j} d_j(\omega_2) d\hat{N}_{2j} + \delta_{2k} d_k(\omega_2) \Delta N_1 - \delta_{2k} d_k(\omega_2) \left(\bar{N}_k - \bar{N}_{k-} \right) - \int_{(k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j \right] \\
=~ &c_1(\omega_1) + \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} \left[\underbrace{\int_{[0,1]} \delta_{2j} d_j(\omega_2) d\hat{N}_{2j}}_{\leq c_2(\omega_2)} - \underbrace{\left(\delta_{2k} d_k(\omega_2) \Delta N_2 + \int_{(k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j \right)}_{= c_2(\omega_2)} \right] \leq c_1(\omega_1),
\end{aligned}
$$

where: the second line follows by feasibility; $\bar{N} = \hat{N}_1 + \hat{N}_2$, the third line follows by rearranging and using the assumption that $(c, \hat{N})$ is incentive feasible; the fourth line follows by using the condition that (45) is strictly increasing; the fifth line by rearranging and using feasibility again; and the sixth line by our assumption that $(c, \hat{N})$ is incentive
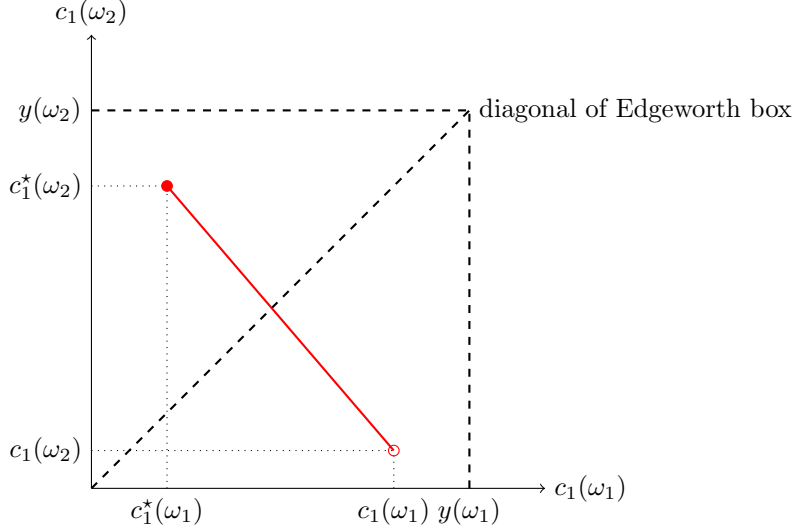
55

Figure 5: The Edgeworth box for the consumption of agent 1 in state ω_1 (x-axis) and in state ω_2 (y-axis).

feasible and by our observation that (21) must hold with equality given our choice of k and ΔN_2.

D.2   Proof of Lemma 3

Consider first the first-best allocation, $c^\star$. The first-order condition of the Planner's problem implies

$$\alpha_1 \left[c_1^\star(\omega)\right]^{-\gamma_1} - \alpha_2 \left[y(\omega) - c_1^\star(\omega)\right]^{-\gamma_2} = 0,$$

for all $\omega \in \Omega$. In terms of consumption share, $c(\omega)/y(\omega)$, this equation becomes:

$$\alpha_1 \left[\frac{c_1^\star(\omega)}{y(\omega)}\right]^{-\gamma_1} y(\omega)^{\gamma_2 - \gamma_1} - \alpha_2 \left[1 - \frac{c_1^\star(\omega)}{y(\omega)}\right]^{-\gamma_2} = 0. \tag{48}$$

Since $\gamma_2 > \gamma_1$, this equation is strictly decreasing in the consumption share and strictly increasing in $y(\omega)$. Hence it follows that the consumption share is strictly increasing in $y(\omega)$, i.e., $c_1^\star(\omega_1)/y(\omega_1) < c_1^\star(\omega_2)/y(\omega_2)$. The inequality for $i = 2$ follows directly because consumption shares add up to one.

Now consider the equilibrium allocation, c. Assume, toward a contradiction, that $c_1(\omega_1)/y(\omega_1) \geq c_1(\omega_2)/y(\omega_2)$, i.e., the consumption shares of agent $i = 1$ lie below the diagonal of the Edgeworth box, as shown in Figure 5. Notice that since the first-best allocation, $c^\star$, satisfies the reverse inequality, it must lie strictly above the diagonal. This implies that $c^\star \neq c$. By strict concavity, social welfare evaluated at c is strictly smaller than social welfare evaluated at $c^\star$, and strictly smaller than social welfare at any point on the segment $(c, c^\star]$ linking c to $c^\star$, shown in red on the figure. Clearly, the segment $[c, c^\star)$ crosses the diagonal at some point $c^\dagger$, which may be c. Since $c^\dagger$ keeps the agent's consumption share constant across states, it can be made incentive feasible by giving agents the corresponding "proportional" tree allocation, i.e., a share in the market portfolio equal to their respective consumption share, $N_i^\dagger = c_i^\dagger(\omega_i)/y(\omega_i) \bar{N}$. But since $\delta < 1$, it follows that all incentive constraints are slack for $(c^\dagger, N^\dagger)$. Therefore, points on the segment $(c, c^\star]$ near $c^\dagger$ are incentive feasible as well. But they improve social welfare strictly relative to c, which is a contradiction.

D.3 Proof of Proposition 6

As for Lemma 2, we offer a proof in the general case when δ_{ij} is assumed to depend both on the identity of the tree holders and on the type of the tree, maintaining the restriction that

$$\frac{\delta_{1j}d_j(\omega_1)}{\delta_{2j}d_j(\omega_2)}, \tag{49}$$

is strictly increasing.

Given Lemma 2, what remains to be shown is that, for any incentive-feasible consumption allocation on the boundary, the distribution of trees is uniquely determined. We establish:

Lemma D.1 *Suppose that (20) and (21) holds with equality for some consumption allocation c, some $k \in [0,1]$ and some $(\Delta N_1, \Delta N_2) \geq 0$ such that $\Delta N_1 + \Delta N_2 = N_k - N_{k-}$. Then (c, N^+) is incentive feasible if and only if $N_1^+ = \Delta N_1 \mathbb{I}_{\{j=k\}} + \bar{N} \mathbb{I}_{\{j<k\}}$ and $N_2^+ = \Delta N_2 \mathbb{I}_{\{j=k\}} + \bar{N} \mathbb{I}_{\{j>k\}}$.*

The "if" part of Lemma D.1 follows because, with the proposed tree allocation, the incentive constraint of agent $i = 1$ binds in state ω_1, and that of agent $i = 2$ binds in state ω_2. It then follows that the two other incentive constraints are slack.

For the "only if" part, consider any tree allocation such that $(c, \hat{N}^+)$ is incentive feasible. Then the incentive constraint of agent $i = 1$ in state ω_1 writes:

$$c_1(\omega_1) = \int_{[0,k)} \delta_{1j}d_j(\omega_1)\, d\bar{N}_j + \delta_{1k}d_k(\omega_1)\Delta N_1 \geq \int \delta_{1j}d_j(\omega_1)d\hat{N}_{1j}^+$$

Using that $d\bar{N}_j = d\hat{N}_{1j}^+ + d\hat{N}_{2j}^+$ we then obtain that:

$$\int_{[0,k)} \delta_{1j}d_j(\omega_1)\, d\hat{N}_{2j}^+ + \delta_{1k}d_k(\omega_1)\Delta N_1 \geq \int_{(k,1]} \delta_{1j}d_j(\omega_1)d\hat{N}_{1j}^+ + \delta_{1k}d_k(\omega_1)\Delta\hat{N}_1, \tag{50}$$

where $\Delta\hat{N}_1 \equiv \hat{N}_{1k} - \hat{N}_{1k-}$. Proceeding analogously with the incentive constraint of agent $i = 2$ in state ω_2, we obtain:

$$\int_{(k,1]} \delta_{2j}d_j(\omega_2)\, d\hat{N}_{1j}^+ + \delta_{2k}d_k(\omega_2)\Delta N_2 \geq \int_{[0,k)} \delta_{2j}d_j(\omega_2)d\hat{N}_{2j}^+ + \delta_{2k}d_k(\omega_2)\Delta\hat{N}_2, \tag{51}$$

where $\Delta\hat{N}_2 \equiv \hat{N}_{2k} - \hat{N}_{2k-}$ Now multiply equation (50) by $\delta_{2k}d_k(\omega_2)$, and equation (51) by $\delta_{1k}d_k(\omega_1)$ and add the two inequalities. The $j = k$ terms cancel each others out because, by feasibility, $\Delta N_1 + \Delta N_2 = \Delta\hat{N}_1 + \Delta\hat{N}_2$. We thus obtain:

$$\int_{[0,k)} \delta_{1j}d_j(\omega_1)\delta_{2k}d_j(\omega_2)\, d\hat{N}_{2j}^+ + \int_{(k,1]} \delta_{2j}d_j(\omega_2)\delta_{1k}d_k(\omega_1)\, d\hat{N}_{1j}^+ \geq \int_{(k,1]} \delta_{1j}d_j(\omega_1)\delta_{2k}d_k(\omega_2)d\hat{N}_{1j}^+ + \int_{[0,k)} \delta_{2j}d_j(\omega_2)\delta_{1k}d_k(\omega_1)d\hat{N}_{2j}^+.$$

After rearranging:

$$\int_{[0,k)} [\delta_{1j}d_j(\omega_1)\delta_{2k}d_j(\omega_2) - \delta_{2j}d_j(\omega_2)\delta_{1k}d_k(\omega_1)]\, d\hat{N}_{2j}^+ \geq \int_{(k,1]} [\delta_{1j}d_j(\omega_1)\delta_{2k}d_k(\omega_2) - \delta_{2j}d_j(\omega_2)\delta_{1k}d_k(\omega_1)]\, d\hat{N}_{1j}^+$$

But, by (49), the integrand on the left-hand side is strictly negative over $[0,k)$, while the integrand on the right-hand side is strictly positive over $(k,1]$. Therefore, both integrals are zero: agent $i = 2$ holds no tree in $[0,k)$ and all trees in $(k,1]$, while agent $i = 1$ holds all trees in $[0,k)$ and no tree in $(k,1]$. Plugging this back into the incentive

compatibility constraint, we can determine each agent's holdings of tree k. Indeed, we obtain:

$$\delta_{1k}d_k(\omega_1)\Delta N_1 \geq \delta_{1k}d_k(\omega_1)\Delta\hat{N}_1 \text{ and } \delta_{2k}d_k(\omega_2)\Delta N_2 \geq \delta_{2k}d_k(\omega_1)\Delta\hat{N}_2.$$

Since $\Delta N_1 + \Delta N_2 = \Delta\hat{N}_1 + \Delta\hat{N}_2$, it follows that $\Delta N_1 = \Delta\hat{N}_1$ and $\Delta N_2 = \Delta\hat{N}_2$.

D.4   Proof of Proposition 7

With two agents, the zero-transfer equation (41) writes:

$$\bar{n}_2\alpha_1\mathbb{E}\left\{u_1'\left[c_1(\omega)\right]c_1(\omega)\right\} = \bar{n}_1\alpha_2\mathbb{E}\left\{u_2'\left[c_2(\omega)\right]c_2(\omega)\right\}$$

With CRRA utility, this can be simplified further:

$$\bar{n}_2\alpha_1\mathbb{E}\left[c_1(\omega)^{1-\gamma_1}\right] = \bar{n}_1\alpha_2\mathbb{E}\left[c_2(\omega)^{1-\gamma_2}\right],$$

so that:

$$\frac{\bar{n}_1}{\bar{n}_2} = \frac{\alpha_1\mathbb{E}\left[c_1(\omega)^{1-\gamma_1}\right]}{\alpha_2\mathbb{E}\left[c_2(\omega)^{1-\gamma_2}\right]}.$$

Now notice that, as α_1/α_2 increases, the solution of the Planner's problem moves to the northeast of the incentive-constrained Pareto set (see the proof of Proposition C.8). Clearly, this implies a strictly increasing relationship between $\bar{n}_1/\bar{n}_2$ and α_1/α_2.

D.5   Proof of Proposition 8

Suppose c is incentive feasible under tree distribution $\bar{N}$ and is such that the consumption share of agent $i = 1$ is higher in state ω_1 than in state ω_2, as is the case in equilibrium (the opposite case is symmetric). Lemma 2 implies that there exist $k \in [0,1]$ and $(\Delta N_1, \Delta N_2) \geq 0$, $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$ such that the incentive compatibility constraints (20) and (21) are satisfied.

To establish the claim of the proposition, that c is incentive feasible under tree distribution $\bar{N}^*$, we need to find some $k^* \in [0,1]$ and $(\Delta N_1^*, \Delta N_2^*) \geq 0$, $\Delta N_1^* + \Delta N_2^* = \bar{N}_{k^*} - \bar{N}_{k^*-}$ such that the analog of conditions (20) and (21) for the new tree allocation (described with variables with a "*" subscript) are satisfied. It is thus sufficient to find some k^* and $(\Delta N_1^*, \Delta N_2^*) \geq 0$, $\Delta N_1^* + \Delta N_2^* = \bar{N}_{k^*} - \bar{N}_{k^*-}$ such that:

$$\int_{j\in[0,k^*)} j d\bar{N}_j^* + k^*\Delta N_1^* \leq \int_{j\in[0,k)} j d\bar{N}_j + k\Delta\hat{N}_1 \tag{52}$$

and

$$\int_{j\in(k^*,1]} (1-j)d\bar{N}_j^* + (1-k^*)\Delta N_2^* \leq \int_{j\in(k,1]} (1-j)d\bar{N}_j + (1-k)\Delta N_2. \tag{53}$$

Let $k^* \in [0,1]$ and $(\Delta N_1^*, \Delta N_2^*) \geq 0$, $\Delta N_1^* + \Delta N_2^* = \bar{N}_{k^*} - \bar{N}_{k^*-}$, such that

$$\bar{N}_{k^*-}^* + \Delta N_1^* = \bar{N}_{k-} + \Delta N_1. \tag{54}$$

First note that (54) together with (23) and (24) imply that (52) holds if and only if (53) holds. Indeed, adding

$$\bar{N}_1^* - \left(\bar{N}_{k^*-}^* + \Delta N_1^*\right) - \int_{j \in [0,1]} j d\bar{N}_j^* = \bar{N}_1 - \left(\bar{N}_{k-} + \Delta N_1\right) - \int_{j \in [0,1]} j d\bar{N}_j$$

on each side of (52) yields (53).

Let us now show that (52) holds. Integrating the left-hand side of (52) by parts yields

$$\int_{j \in [0,k^*)} j d\bar{N}_j^* + k^* \Delta N_1^* = k^* \bar{N}_{k^*-}^* - \int_0^{k^*} \bar{N}_j^* dj + k^* \Delta N_1^*.$$

There are two cases to consider.

First case: $k^* \le k$. Then, it follows from (54) that $\bar{N}_{k^*-}^* + \Delta N_1^* = \bar{N}_{k-} + \Delta N_1 \ge \bar{N}_{k-} \ge \bar{N}_j$ for $j \in (k^*, k)$. Thus,

$$
\begin{aligned}
\int_{j \in [0,k^*)} j d\bar{N}_j^* + k^* \Delta N_1^* &\le k^* \bar{N}_{k^*-}^* - \int_0^{k^*} \bar{N}_j^* dj + k^* \Delta N_1^* + \int_{k^*}^k (\bar{N}_{k^*-}^* + \Delta N_1^* - \bar{N}_j) dj \\
&= k \left(\bar{N}_{k^*-}^* + \Delta N_1^*\right) - \int_0^{k^*} \bar{N}_j^* dj - \int_{k^*}^k \bar{N}_j dj \\
&= k \left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j^* dj - \int_{k^*}^k \bar{N}_j dj \\
&\le k \left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j dj - \int_{k^*}^k \bar{N}_j dj \\
&= k \left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^k \bar{N}_j dj
\end{aligned}
$$

where the second line is obtained by calculation, the third line follows from (54), the fourth line follows from the definition of second order stochastic dominance, the fifth line is obtained by calculation and is equal to the right-hand side of (52) after integration by parts.

Second case: $k < k^*$. Then

$$
\begin{aligned}
\int_{j \in [0,k^*)} j d\bar{N}_j^* + k^* \Delta N_1^* &= k^* \left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j^* dj \\
&\le k^* \left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j dj \\
&\le k^* \left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^k \bar{N}_j dj - \int_k^{k^*} \bar{N}_k dj \\
&\le k^* \left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^k \bar{N}_j dj - (k^* - k) \left(\bar{N}_{k-} + \Delta N_1\right) \\
&= k \left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^k \bar{N}_j dj
\end{aligned}
$$

where the first line follows by integration by part of (54), the second line follows from the definition of second order stochastic dominance, the third line follows from $\bar{N}$ being increasing, the fourth line follows from $\Delta N_2 \ge 0$, the fifth line is obtained by calculation and is equal to the right-hand side of (52) after integration by parts.

D.6 Proof of Proposition 10

Notice that, since the function ϕ_ℓ is the same for both agents, we have that $\delta_{1j}d_j(\omega_1)/\delta_{2j}d_j(\omega_2) = d_j(\omega_1)/d_j(\omega_2)$ is strictly increasing, so all our results apply.

The equilibrium is uniquely pinned down by a two-equation-in-two-unknown problem, for the ratio of the two budget constraints multipliers, $r = \frac{\lambda_1}{\lambda_2}$, and the threshold k determining tree ownership. To obtain the first equation, first note that the continuity of $j \mapsto (\delta_{1j}d_j(\omega_1))/(\delta_{2j}d_j(\omega_2))$ implies that for the threshold tree, the first-order condition with respect to tree holdings holds with an equality for both agents. Thus:

$$F(r,k) \equiv \mu_1(\omega_1)\delta_{1k}d_k(\omega_1) - r\mu_2(\omega_2)\delta_{2k}d_k(\omega_2) = 0. \tag{55}$$

where, from the first-order conditions we have that

$$\mu_1(\omega_1) = r\pi(\omega_1)u_2'\left[\int_0^1 \left(1 - \delta_{1j}\mathbb{I}_{\{j<k\}}\right)d_j(\omega_1)\,d\bar{N}_j\right] - \pi(\omega_1)u_1'\left[\int_0^1 \delta_{1j}\mathbb{I}_{\{j<k\}}d_j(\omega_1)\,d\bar{N}_j\right]$$

$$\mu_2(\omega_2) = \frac{1}{r}\pi(\omega_2)u_1'\left[\int_0^1 \left(1 - \delta_{2j}\mathbb{I}_{\{j\geq k\}}\right)d_j(\omega_2)\,d\bar{N}_j\right] - \pi(\omega_2)u_2'\left[\int_0^1 \delta_{2j}\mathbb{I}_{\{j\geq k\}}d_j(\omega_2)\,d\bar{N}_j\right].$$

Notice that the continuity of the distribution of tree supplies mean that there is no atom, hence $\Delta N_1 = \Delta N_2 = 0$, i.e., the allocation of the supply of threshold tree between agents is irrelevant. The second equilibrium equation is (41) which here takes the form:

$$G(r,k) \equiv \mathbb{E}[u_1'(c_1(\omega))c_1(\omega)] - r\frac{\bar{n}_1}{\bar{n}_2}\mathbb{E}[u_2'(c_2(\omega))c_2(\omega)] = 0, \tag{56}$$

where $c_1(\omega_1) = \int_0^k \delta_{1j}d_j(\omega_1)\,d\bar{N}_j$, $c_2(\omega_1) = \int_0^1 d_j(\omega_1)\,d\bar{N}_j - c_1(\omega_1)$, $c_2(\omega_2) = \int_k^1 \delta_{2j}d_j(\omega_2)\,d\bar{N}_j$, and $c_1(\omega_2) = \int_0^1 d_j(\omega_2)\,d\bar{N}_j - c_2(\omega_2)$.

The function $F(r,k)/(\delta_{2k}d_k(\omega_2))$ is strictly increasing and continuous in both r and k. Moreover, one can explicitly solve for r as a function of k, $\rho(k)$. This function is strictly decreasing and, because of the Inada condition $u_i'(0) = +\infty$, goes to infinity as k goes to zero, $\lim_{k\to 0}\rho(k) = \infty$, and goes to zero as k goes to one, $\lim_{k\to 1}\rho(k) = 0$.

Since $\bar{N}_j$ is strictly increasing, it follows that both $c_1(\omega_1)$ and $c_1(\omega_2)$ are strictly increasing in k while both $c_2(\omega_1)$ and $c_2(\omega_2)$ are strictly decreasing in k. Recall that the coefficient of relative risk aversion are both less than one, $0 \leq \gamma_1 < \gamma_2 \leq 1$. Therefore, the function $G(r,k)$ is strictly decreasing in r and strictly increasing in k. Plugging in the function $\rho(k)$ defined above, we obtain a strictly increasing function $k \mapsto G(\rho(k),k)$. Given our earlier observation that $\lim_{k\to 0}\rho(k) = \infty$ and $\lim_{k\to 1}\rho(k) = 0$, it follows that $k \mapsto G(\rho(k),k)$ is strictly negative when $k \simeq 0$, and strictly positive when $k \simeq 1$. Thus, the equilibrium threshold is the unique solution of $G(\rho(k),k) = 0$. Clearly $c_1(\omega_1)$ increases with ε, while $c_2(\omega_2)$ stays the same. This implies that $\rho(k)$ shifts down, and that $G(\rho(k),k)$ shifts down as well. Hence $k(\varepsilon') < k(\varepsilon)$ if $\varepsilon' > \varepsilon$.

$$\frac{dk}{d\varepsilon} < 0.$$

E Supplementary appendix

E.1 Proof of Lemma C.1

For this proof, in order to apply some of the results in Chapter 12 of Stokey and Lucas (1989), we extend measures $M^+ \in \mathcal{M}_+$ to the entire real line, $\mathbb{R}$, by setting $M_j^+ = 0$ for all $j < 0$, and $M_j^+ = M_1^+$ for all $j \geq 1$. Now consider a sequence (c^k, N^{k+}), $k \in \mathbb{N}$, of incentive feasible allocations. Given that c^k belongs to a finite dimensional space

and is bounded, it has a converging subsequence. Given that $\sum_{i\in I} N_i^{k+} = \bar{N}$, N_{ij}^+ is bounded above by $\bar{N}_j$ for all $(i,j) \in I \times \mathbb{R}$, an an application of Helly's selection Theorem (Theorem 12.9 in Stokey and Lucas (1989) easily extended to finite measures instead of distributions) shows that for each $i \in I$, N_i^{k+} has a subsequence such that $N_i^{\ell+}$ converging weakly in $\mathcal{M}_+$. Taken together, this means that there exists a subsequence $(c^\ell, N^{\ell+})$ of (c^k, N^{k+}) and some (c, N^+) such that $c^\ell \to c$ and $N_i^{\ell+} \Rightarrow N_i^+$ for each $i \in I$.

What is left to show is that (c, N^+) is incentive feasible. Given that $j \mapsto d_j(\omega)$ and $j \mapsto \delta_{ij}$ are continuous, the definition of weak convergence allows us to assert that, since the feasibility constraint for consumption, (9), and the incentive compatibility constraints, (38), hold for each $(c^\ell, N^{\ell+})$, then it must also hold in the limit for (c, N^+). The only difficulty is to show that the feasibility constraint for holdings is also satisfied. For this we rely on the characterization of weak convergence provided by Theorem 12.8 in Stokey and Lucas (1989), easily extended to finite measures. It asserts that $N_i^{\ell+}$ converges pointwise at each continuity point of their limit, N_i^+. Therefore, for any $j \in \mathbb{R}$ such that all $(N_i^+)_{i\in I}$ are continuous, we have:

$$\sum_{i\in I} N_{ij}^{\ell+} \to \sum_{i\in I} N_{ij}^+.$$

But recall that the feasibility constraint for holdings is satisfied for each j: $\sum_{i\in I} N_{ij}^{\ell+} = \bar{N}_j$. Together with the above, this implies that

$$\sum_{i\in I} N_{ij}^+ = \bar{N}_j,$$

for all $j \in \mathbb{R}$ such as all $(N_i^+)_{i\in I}$ are continuous. Now recall that the N_i^+'s are increasing functions, and so have countably many discontinuity points. This implies that for any $j \in \mathbb{R}$, there is a sequence of $j_n \downarrow j$ such that j_n is a continuity point for all $(N_i^+)_{i\in I}$. Hence, for all j_n, we have

$$\sum_{i\in I} N_{ij_n}^+ = \bar{N}_{j_n}^+.$$

Since $j \mapsto N_{ij}^+$ and $\bar{N}_j$ are all right continuous functions we can take the limit $j_n \downarrow j$ and obtain that $\sum_{i\in I} N_{ij}^+ = \bar{N}_j$ for all $j \in \mathbb{R}$, as required.

E.2   Proof of Proposition C.1

In all what follow we let:
$$y(\omega) \equiv \int d_j(\omega)\, d\bar{N}_j, \ \underline{y} \equiv \min_{\omega\in\Omega} y(\omega), \text{ and } \bar{y} \equiv \max_{\omega\in\Omega} y(\omega).$$

Proof that $\Gamma^\star(\alpha)$ is not empty.   We first show that the supremum is achieved. The only difficulty with this proof arises when $\alpha_i > 0$ and $u_i(0) = -\infty$ for some $i \in I$, because in this case the objective is unbounded as $c_i(\omega) \to 0$. However, in the planner's problem, one can restrict attention to $c_i(\omega)$ that are bounded away from zero. To see this, we first note that $c_h(\omega) = y(\omega)/I$, $N_h+ = \bar{N}/I$, for all $h \in I$, is an incentive-feasible allocation, implying that:

$$W^\star(\alpha) \geq \sum_{i\in I} \alpha_i \sum_{\omega\in\Omega} \pi(\omega) u_i\left[y(\omega)/I\right] \geq \underline{W}_i(\alpha) \text{ where } \underline{W}_i(\alpha_i) \equiv \alpha_i \pi(\omega) u_i(\underline{y}/I) + \sum_{h\neq i} \min\left\{ u_h\left(\underline{y}/I\right), 0\right\}.$$

Also, for any allocation (c, N^+), we have that

$$W(\alpha, c, N^+) \leq \alpha_i \pi(\omega) u_i\left[c_i(\omega)\right] + \sum_{h\neq i} \max\{u_h(\bar{y}), 0\},$$

for any $i \in I$. Now consider the equation

$$\alpha_i \pi(\omega) u_i \left[c_i(\omega) \right] + \sum_{h \neq i} \max\{u_k(\bar{y}), 0\} = \underline{W}_i(\alpha_i, \omega) = \alpha_i \pi(\omega) u_i(\underline{y}/I) + \sum_{h \neq i} \min\left\{ u_k \left(\underline{y}/I \right), 0 \right\},$$

for some fixed i such that $\alpha_i > 0$ and $u_i(0) = -\infty$. Since $u_i(0) = -\infty$, the left-hand side is smaller than the right-hand side when $c \to 0$. Since $u_i(c)$ is strictly increasing, the left-hand side is greater than the right-hand side when $c > \underline{y}/I$, it follows that the equation has a unique solution, which is less than $\underline{y}/I$. Clearly, the solution is increasing and continuous in α_i. Let $\underline{c}_i(\alpha_i)$ be half of the minimum of these solutions across all $\omega \in \Omega$. By construction, for any allocation (c, N^+) such that $c_i(\omega) < \underline{c}_i(\alpha_i)$ for some $\omega \in \Omega$, $W(\alpha, c, N^+)$ is strictly less than the value attained by $c_h(\omega) = y(\omega)/I$ and $N_h^+ = \bar{N}/I$, and so cannot be optimal. If we let $\underline{c}_i(\alpha_i) = 0$ for other i, that is for $i \in I$ such that $\alpha_i = 0$ or $u_i(0) = 0$, then, in the Planner's problem, one can restrict attention to allocation such that $c_i(\omega) \geq \underline{c}_i(\alpha_i)$, which we write as $c \geq \underline{c}(\alpha)$. Notice that, by construction, the objective of the planner is continuous over $c \geq \underline{c}(\alpha)$.

Now to show that there is a solution consider any sequence (c^k, N^{k+}) of incentive-feasible allocation such that $W(\alpha, c^k, N^{k+}) \to W^\star(\alpha)$. From the above remark we can focus on a sequence such that $c^k \geq \underline{c}(\alpha)$. Now, by Lemma C.1, there exists some incentive feasible allocation (c, N^+) and a subsequence $(c^\ell, N^{\ell+})$ such that $c^\ell \to c$ and $N^{\ell+} \to N^+$. Going to the limit in the Planner's objective, we obtain that $W(\alpha, c, N^+) = W^\star(\alpha)$.

Proof that $\Gamma^\star(\alpha)$ is weakly compact. The argument is the same as in the last paragraph, except that we now consider a sequence $(c^k, N^{k+}) \in \Gamma^\star(\alpha)$.

Proof that $\Gamma^\star(\alpha)$ convex-valued. This follows because the objective is concave and the constraints linear.

Proof that $W^\star(\alpha)$ is continuous and $\Gamma^\star(\alpha)$ has a weakly closed graph. Consider any $\bar{\alpha} \geq 0$ such that $\sum_{i \in I} \bar{\alpha}_i = 1$ and any sequence $\alpha^k \to \bar{\alpha}$ and an associated sequence $(c^k, N^{k+}) \in \Gamma^\star(\alpha^k)$. Without loss of generality for this proof, assume that $W^\star(\alpha^k)$ converges to some limit,[25] and that (c^k, N^{k+}) converges weakly towards some incentive feasible allocation (c, N^+).[26] We want to show that $W^\star(\alpha^k) \to W^\star(\alpha)$ and that $(c, N^+) \in \Gamma^\star(\alpha)$. Let $I_0 = \{i \in I : \bar{\alpha}_i = 0 \text{ and } u_i(0) = -\infty\}$. We have:

$$W^\star(\alpha^k) = \sum_{i \notin I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega) \right] + \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega) \right]. \tag{57}$$

By our maintained assumptions, both the left-hand side and the first term on the right-hand side have a limit as $k \to \infty$. Hence, the second term on the right-hand side has a limit as well. We argue that this limit must be negative. Indeed, for $i \in I_0$, if $\lim c_i^k(\omega) > 0$, then $\lim \alpha_i^k u_i \left[c_i^k(\omega) \right] = 0$. If $\lim c_i^k(\omega) = 0$, then $\alpha_i^k u_i \left[c_i^k(\omega) \right] \leq 0$ for k large enough. Hence,

$$\lim \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega) \right] \leq 0.$$

Therefore:

$$\lim W^\star(\alpha^k) \leq \sum_{i \notin I_0} \bar{\alpha}_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[\lim c_i^k(\omega) \right] \leq W^\star(\bar{\alpha}), \tag{58}$$

since $(\lim c^k, \lim N^k)$ is incentive feasible.

To show the reverse inequality, for all $i \in I_0$, choose some $\phi_i > 0$ such that $1 + \phi_i(1 - \gamma_i) > 0$, where $\gamma_i > 1$ is the assumed CRRA bound for $u_i(c)$. Let $\beta(\alpha) \equiv \sum_{i \in I_0} (\alpha_i)^{\phi_i}$. Since $\lim \alpha_i^k = 0$ for all $i \in I_0$, we have that $\lim \beta(\alpha^k) = 0$, hence $\beta(\alpha^k) < 1$ for all k large enough. Now take any $(c, N^+) \in \Gamma^\star(\bar{\alpha})$. For all $i \in I_0$ we have that $\bar{\alpha}_i = 0$, which clearly implies that $c_i(\omega) = N_i^+ = 0$, i.e., $i \notin I_0$ consume the aggregate endowment and holds the

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[25]Indeed, since $W^\star(\alpha)$ is bounded below by $\min\{\underline{W}_i(\alpha_i, \omega), i \in I, \alpha_i \in [0, 1], \omega \in \Omega\}$ and is clearly bounded above, to show convergence towards $W^\star(\alpha)$ it is sufficient to show that every convergent subsequence of $W^\star(\alpha^k)$ converges towards $W^\star(\alpha)$.

[26]From Lemma C.1, we can always find a convergence subsequence with this property.

aggregate tree supplies. Therefore, if we scale down the consumption and tree holding of $i \notin I_0$ by $1 - \beta(\alpha^k)$, we keep the allocation of $i \notin I_0$ incentive compatible and we free up $\beta(\alpha^k)y(\omega)$ consumption, and $\beta(\alpha^k)\bar{N}$ trees. We can then re-distribute this consumption and these trees by giving to each agent $i \in I_0$ a consumption equal to $y(\omega)\left(\alpha_i^k\right)^{\phi_i}$ and a tree allocation equal to a fraction $\left(\alpha_i^k\right)^{\phi_i}$ of the market portfolio, $\bar{N}$. Because the consumption of $i \in I_0$ is proportional to its portfolio payoff, the allocation of $i \in I_0$ is incentive compatible. Therefore, this process of scaling down the consumption of $i \notin I_0$ and redistributing to $i \in I_0$, leads to an incentive feasible allocation. Hence, we have that:

$$W^\star(\alpha^k) \geq \sum_{i \notin I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i(\omega)(1 - \beta(\alpha^k)) \right] + \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[y(\omega) \left(\alpha_i^k \right)^{\phi_i} \right].$$

The first term converges to $W^\star(\bar{\alpha})$. Using the assumed CRRA bound, $0 < |u(c)| < |K|c^{1-\gamma_i}$ for c close to zero, one sees that the second term goes to zero: indeed $\alpha_i^k \left| u_i \left[y(\omega) \left(\alpha_i^k \right)^{\phi_i} \right] \right|$ is bounded above by $|K| y(\omega)^{1-\gamma_i} \left(\alpha_i^k \right)^{1+(1-\gamma_i)\phi_i}$, which goes to zero since $\lim \alpha_i^k = 0$ and ϕ_i was chosen so that $1 + \phi_i(1-\gamma_i) > 0$. Hence, we obtain that $\lim W^\star(\alpha^k) \geq W^\star(\bar{\alpha})$. Taken together we have that

$$\lim W^\star(\alpha^k) \geq \sum_{i \notin I_0} \bar{\alpha}_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[\lim c_i^k(\omega) \right] = W^\star(\bar{\alpha}). \tag{59}$$

Taken together, (58) and (59) imply that

$$\lim W^\star(\alpha^k) = \sum_{i \notin I_0} \alpha_i \sum_{\omega \in \Omega} u_i \left[\lim c_i^k(\omega) \right] = W(\bar{\alpha}) \text{ and } \lim \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega) \right] = 0.$$

This establishes that $W^\star(\alpha)$ is continuous and that $\Gamma^\star(\alpha)$ has a closed graph.

Proof that $\lim \alpha_i^k u' \left[c_i^k(\omega) \right] c_i^k(\omega) = 0$ **if** $\lim \alpha_i^k = 0$. Consider any sequence $\alpha^k \to \bar{\alpha}$ and any associated sequence (not necessarily converging) (c^k, N^{k+}) in $\Gamma^\star(\alpha^k)$. Since we have shown that $\Gamma^\star(\alpha)$ has a weakly closed graph, it follows that any converging subsequence of (c^k, N^{k+}) has a limit belonging to $\Gamma^\star(\bar{\alpha})$. Since the Planner finds optimal to give zero consumption to agents with zero weight, it follows that $\lim c_i^k(\omega) = 0$ for all i such that $\bar{\alpha}_i = 0$.

If $u_i(0) = 0$, then the result that $\lim \alpha_i^k u' \left[c_i^k(\omega) \right] c_i^k(\omega) = 0$ follows from the inequality $0 \leq u_i'(c)c \leq u_i(c)$. If $u_i(0) = -\infty$, we need a different argument. Write $W^\star(\alpha^k) = W_1^k + W_2^k$, where

$$W_1^k \equiv \sum_{i \notin I_0} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega) \right] \text{ and } W_2^k \equiv \sum_{i \in I_0} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega) \right].$$

By assumption, we have that $\lim \left(W_1^k + W_2^k \right) = W^\star(\bar{\alpha})$. Notice that W_1^k is bounded. Indeed, it is clearly bounded above because the constraint set is bounded. It is bounded below because, for any $i \notin I_0$ such that $u_i(0) = -\infty$, $\bar{\alpha}_i > 0$ and so α_i^k and hence $\underline{c}_i(\alpha_i^k)$ is bounded away from zero for k large enough. Given boundedness, we can extract some convergent subsequence W_1^ℓ of W_1^k. Since consumption and tree holdings are incentive feasible, it follows from Lemma C.1 that there exists a weakly convergent subsequence (c^p, N^{p+}) of $(c^\ell, N^{\ell+})$. Clearly, $\lim W_1^p = \lim W_1^\ell$. But, using the results of the previous paragraph, we have that $\lim W_1^p = W^\star(\bar{\alpha})$. Hence all convergent subsequences of W_1^k have the same limit $W^\star(\bar{\alpha})$, implying that $\lim W_1^k = W^\star(\bar{\alpha})$ and that $\lim W_2^k = 0$. It follows that, asymptotically as $k \to \infty$, the aggregate consumption of agents $i \notin I_0$ is arbitrarily close to $y(\omega)$, and the consumption of each agent $i \in I_0$ is arbitrarily close to zero. Therefore, for all k large enough, all terms in W_2^k are negative. Hence, for k large enough, we that for all $i \in I_0$, $W_2^k \leq \alpha_i^k \pi(\omega) u_i \left[c_i^k(\omega) \right] \leq 0$. Since $\lim W_2^k = 0$, it follows that $\lim \alpha_i^k \pi(\omega) u_i \left[c_i^k(\omega) \right] = 0$ as well. The result then follows from the CRRA bound $0 \leq u_i'(c)c \leq \gamma_i |u_i(c)|$.

E.3   Proof of Proposition C.2

Fix any feasible $N^+ \in \mathcal{M}_+$ and let:

$$W(\alpha \,|\, N^+) = \max \sum_{i \in I} \alpha_i U_i(c_i)$$

with respect to $c \in \mathbb{R}_+^{|\Omega| \times |I|}$, and subject to

$$\sum_{i \in I} c_i(\omega) \leq \sum_{i \in I} \int d_j(\omega) dN_{ij}^+ \quad \forall \omega \in \Omega$$

$$c_i(\omega) \geq \int \delta_{ij} d_j(\omega) dN_{ij}^+ \quad \forall (i,\omega) \in I \times \Omega.$$

From Corollary 28.3 in Rockafellar (1970), $c \in \mathbb{R}_+^{|\Omega| \times |I|}$ is an optimal solution only if there exists multipliers $\hat{q} \in \mathbb{R}_+^{|\Omega|}$ and $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$ such that:

$$\alpha_i \frac{\partial U_i}{\partial c_i(\omega)} + \hat{\mu}_i(\omega) \quad \leq \quad \hat{q}(\omega)$$

$$\hat{q}(\omega) \left[\sum_{i \in I} \int d_j(\omega) \, dN_{ij}^+ - \sum_{i \in I} c_i(\omega) \right] \quad = \quad 0, \quad \forall \omega \in \Omega$$

$$\hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) \, dN_{ij}^+ \right] \quad = \quad 0, \quad \forall (i,\omega) \in I \times \Omega.$$

Notice that there exists multipliers such that the top first-order condition, with respect to $c_i(\omega)$, holds with equality. Indeed, if it holds with a strict inequality for some $\hat{\mu}_i(\omega)$ and some (i,ω), then $c_i(\omega) = 0$ and so the incentive constraint holds with equality. So increasing $\hat{\mu}_i(\omega)$ leaves the complementary slackness conditions unchanged.

Now consider any other feasible $\hat{N}^+ \in \mathcal{M}_+$. Clearly, for any $h \in [0,1]$, $(1-h)N^+ + h\hat{N}^+ = N^+ + h(\hat{N}^+ - N^+)$ is also feasible. In the optimization problem associated with $W\left(\alpha \,|\, N^+ + h \left[\hat{N}^+ - N^+ \right] \right)$, we take the derivative of the Lagrangian with respect to h, and we evaluate this derivative at $h = 0$, given some optimal consumption for $W(\alpha \,|\, N^+)$ and Lagrange multipliers satisfying the first order-conditions. We obtain:

$$L_h \quad = \quad \sum_{i \in I} \int \left[\sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega) \right] \left[d\hat{N}_{ij}^+ - dN_{ij}^+ \right]$$

$$= \quad \sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}_{ij}^+ - dN_{ij}^+ \right],$$

where, for any set of Lagrange multipliers, $\hat{v}_{ij} \equiv \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega)$. Notice that $\hat{q}(\omega)$ is uniquely determined by the first-order conditions[27] but $\hat{\mu}_i(\omega)$ may not, when $c_i(\omega) = 0$. One easily sees in particular that any

$$0 \leq \hat{\mu}_i(\omega) \leq \hat{q}(\omega) - \alpha_i \frac{\partial U_i}{\partial c_i(\omega)}$$

solves the first-order conditions. Let $\hat{V}$ denote the set of $\hat{v}_{ij}$ that is generated by all $\hat{q}$ and $hat\mu_i(\omega)$ solving the first-order conditions. It follows from Corollary 5 in Milgrom and Segal (2002) that the right-

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[27]Indeed for any $\omega \in \Omega$, there exists some $i \in I$ such that the incentive compatibility constraint does not bind. For this $i \in I$, $c_i(\omega) > 0$ and so the first-order condition holds with equality. If $u_i(c)$ is linear, then $\alpha_i \partial U_i / \partial c_i(\omega) = \alpha_i$ is uniquely determined. If $u_i(c)$ is strictly concave, then $c_i(\omega)$ is uniquely determined and so is $\alpha_i \partial U_i / \partial c_i(\omega)$. Using the first-order condition, it then follows that $\hat{q}(\omega)$ is uniquely determined.

derivative of $W\left(\alpha \mid N^+ + h\left[\hat{N}^+ - N^+\right]\right)$ at $h = 0$ is

$$\frac{d}{dh}W\left(\alpha \mid N^+ + h\left[\hat{N}^+ - N^+\right]\right)\bigg|_{h=0^+} = \min_{\hat{v} \in \hat{V}} \sum_{i \in I} \int \hat{v}_{ij}\left[d\hat{N}_{ij}^+ - dN_{ij}^+\right].$$

Next we determine which $\hat{v} \in \hat{V}$ achieves the minimum above. We first notice that $\int \hat{v}_{ij} dN_{ij}^+$ does not depend on the particular choice of $\hat{v} \in \hat{V}$. Indeed, whenever $\hat{v}_{ij}$ is not uniquely determined, it is because $c_i(\omega) = 0$ for some $\omega \in \Omega$. But from the incentive compatibility constraint, it then follows that $\int \delta_{ij} d_j(\omega)\, dN_{ij}^+ = 0$, so $\hat{\mu}_i(\omega)\int \delta_{ij} d_j(\omega)\, dN_{ij}^+ = 0$ as well and $\hat{v}_{ij} = \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega)$, which is uniquely determined. Now, since $\hat{N}_{ij}^+$ is a positive measure, $\int \hat{v}_{ij} d\hat{N}_{ij}^+$ is minimized when $\hat{v}_{ij}$ is smallest, which occurs when $\hat{\mu}_i(\omega)$ is largest, that is, when it is chosen so that the first-order condition with respect to $c_i(\omega)$ holds with equality.

Taken together, we obtain that a necessary condition for a feasible N to be optimal is that:

$$\sum_{i \in I} \int \hat{v}_{ij}\left[d\hat{N}_{ij}^+ - dN_{ij}^+\right] \le 0, \tag{60}$$

for all feasible $\hat{N}^+$, where $\hat{v}_{ij} = \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega)\delta_{ij} d_j(\omega)$ and $\hat{\mu}_i(\omega)$ is chosen so that the first-order condition with respect to $c_i(\omega)$ holds with equality. The proof is concluded by the following Lemma:

Lemma E.1 *Condition (60) holds if and only if $\int \left[\max_{k \in I} \hat{v}_{kj} - \hat{v}_{ij}\right] dN_{ij}^+ = 0$ for all $i \in I$.*

For necessity, consider the correspondence $\Gamma(j) \equiv \arg\max_{k \in I} \hat{v}_{kj}$. By the Measurable Selection Theorem (Theorem 7.6 in Stokey and Lucas (1989)), there exists a measurable selection $\gamma(j)$. Consider then the tree allocation:

$$\hat{N}_{ij}^+ = \int_0^j \mathbb{I}_{\{\gamma(k)=i\}} d\bar{N}_k,$$

which gives the supply of tree k to one agent with the highest valuation, $v_{\gamma(k)k}$. Condition (60) implies that:

$$\begin{aligned}
0 \ge \sum_{i \in I} \int \hat{v}_{ij}\left[d\hat{N}_{ij}^+ - dN_{ij}^+\right] &= \sum_{i \in I} \int \hat{v}_{ij}\mathbb{I}_{\{\gamma(j)=i\}}\, d\bar{N}_j - \sum_{i \in I} \hat{v}_{ij}\, dN_{ij}^+ \\
&= \int \max_{k \in I} \hat{v}_{kj} d\bar{N}_j - \sum_{i \in I} \int \hat{v}_{ij} dN_{ij}^+ \\
&= \sum_{i \in I} \int \left(\max_{k \in I} \hat{v}_{kj} - \hat{v}_{ij}\right) dN_{ij}^+,
\end{aligned}$$

where the second equality follows because $\sum_{i \in I} \hat{v}_{ij}\mathbb{I}_{\{\gamma(j)=i\}} = \max_{k \in I} \hat{v}_{kj}$, and the third equality follows because $\bar{N} = \sum_{i \in I} N_i$. But each term in the sum is positive since $\max \hat{v}_{kj} - \hat{v}_{ij} \ge 0$. It thus follows that each term in the sum is zero, and we are done.

For sufficiency, write

$$\begin{aligned}
\sum_{i \in I} \int \hat{v}_{ij}\left[d\hat{N}_{ij}^+ - dN_{ij}^+\right] &= \sum_{i \in I} \int \hat{v}_{ij}\, d\hat{N}_{ij}^+ - \sum_{i \in I} \int \max_{k \in I} v_{kj} dN_{ij}^+ \\
&= \sum_{i \in I} \int \hat{v}_{ij}\, d\hat{N}_{ij}^+ - \int \max_{k \in I} v_{kj} d\bar{N}_j \\
&= \sum_{i \in I} \int \left[\hat{v}_{ij} - \max_{k \in I} v_{kj}\right] d\hat{N}_{ij}^+ \le 0.
\end{aligned}$$

where the last equality follows because $\hat{N}^+$ is feasible, so $\bar{N}_j = \sum_{i \in I} \hat{N}_{ij}^+$.

E.4 Proof of Proposition C.3

Consider any (c, N^+) and multipliers $\hat{q}$, $\hat{\mu}$ and $\hat{p}$ satisfying the first-order conditions in the Proposition. Now let $(\hat{c}, \hat{N}^+)$ denote any other feasible allocation. We have:

$$
\sum_{i \in I} \alpha_i U_i(c_i) - \sum_{i \in I} \alpha_i U_i(\hat{c}_i)
$$

$$
\geq \quad \sum_{i \in I} \sum_{\omega \in \Omega} \frac{\partial U_i}{\partial c_i(\omega)} \left[c_i(\omega) - \hat{c}_i(\omega) \right] = \sum_{i \in I} \sum_{\omega \in \Omega} \left[\hat{q}(\omega) - \hat{\mu}_i(\omega) \right] \left[c_i(\omega) - \hat{c}_i(\omega) \right]
$$

$$
= \quad \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\sum_{i \in I} c_i(\omega) - \sum_{i \in I} \int d_j(\omega) \, dN_{ij}^+ \right] - \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\sum_{i \in I} \hat{c}_i(\omega) - \sum_{i \in I} \int d_j(\omega) \, d\hat{N}_{ij}^+ \right]
$$

$$
- \sum_{i \in I} \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) dN_{ij}^+ \right] + \sum_{i \in I} \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \left[\hat{c}_i(\omega) - \int \delta_{ij} d_j(\omega) d\hat{N}_{ij}^+ \right]
$$

$$
+ \sum_{i \in I} \int \hat{v}_{ij} \left[dN_{ij}^+ - d\hat{N}_{ij}^+ \right] \geq \sum_{i \in I} \int \hat{v}_{ij} \left[dN_{ij}^+ - d\hat{N}_{ij}^+ \right],
$$

where the last inequality follows from the complementarity slackness for (c, N^+), and from the feasibility of $(\hat{c}, \hat{N})$. Now since both N^+ and $\hat{N}^+$ are feasible, we have that:

$$
\hat{p} \cdot \bar{N} = \hat{p} \cdot \sum_{i \in I} N_{ij} = \hat{p} \cdot \sum_{i \in I} \hat{N}_{ij}.
$$

Hence, adding and subtracting $\hat{p} \cdot \bar{N}$, we obtain:

$$
\sum_{i \in I} \int \hat{v}_{ij} \left[dN_{ij}^+ - d\hat{N}_{ij}^+ \right] = \sum_{i \in I} \left[\hat{p} \cdot \hat{N}_{ij}^+ - \int \hat{v}_{ij} \, d\hat{N}_{ij} \right] - \sum_{i \in I} \left[\hat{p} \cdot N_{ij}^+ - \int \hat{v}_{ij} \, dN_{ij}^+ \right] \geq 0
$$

where the last inequality follows from the first-order condition with respect to N^+.

E.5 Proof of Proposition C.5

Necessity. let (c, N^+) and (p, q) be an equilibrium. Since $\bar{n}_i > 0$, it follows from the first-order conditions of the agent's problem that $\lambda_i > 0$. By direct comparison of first-order conditions, one can then verify that the equilibrium allocation solves the Planner's Problem with weights

$$
\alpha_i = \frac{1/\lambda_i}{\sum_{k \in I} 1/\lambda_k}.
$$

The associated Lagrange multipliers are $\hat{\mu}_i(\omega) = \alpha_i \mu_i(\omega)$, $\hat{q}(\omega) = \beta q(\omega)$ and $\hat{v}_{ij} = \beta v_{ij}$ and $\hat{p} = \beta p$, where $\beta \equiv \left[\sum_{i \in I} 1/\lambda_i \right]^{-1}$. Finally, we have from Lemma C.2 that:

$$
\alpha_i \sum_{\omega \in \Omega} \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) = \bar{n}_i \int \hat{p}_j \, d\bar{N}_j.
$$

Adding up across all $i \in I$ and using $\sum_{i \in I} \bar{n}_i = 1$ yields the desired condition.

Sufficiency. Consider any solution of the Planner's problem satisfying the conditions stated in the Proposition. Notice that the second condition implies that $\alpha_i > 0$. Using Proposition C.2 we obtain associated multipliers $\hat{q}$, $\hat{\mu}$ and $\hat{p}$. We do not assume here that $\hat{p}$ has a dot product representation: as in Proposition C.2 we only assume that $\hat{p}$ is a continuous linear functional. Consider then the candidate equilibrium prices $q(\omega) = \hat{q}(\omega)$ and $p = \hat{p}$, where the definition of equilibrium is extended in the obvious way when $\hat{p}$ does not have a dot product representation. Then, by direct comparison of first-order conditions, one sees that the component (c_i, N_i^+) of the Planner's allocation solves the necessary and sufficient conditions of agent $i \in I$'s problem, in Proposition C.4, except perhaps for the budget feasibility condition and the associated complementary slackness condition. The associated multipliers are $\lambda_i = 1/\alpha_i$, $\mu_i(\omega) = \hat{\mu}_i(\omega)/\alpha_i$ and $v_{ij} = \hat{v}_{ij}$. Therefore, to complete the proof, we need to verify that (c_i, N_i^+) satisfies budget feasibility. For this we calculate the gap between the left- and the right-hand sides of the budget constraint:

$$
\begin{aligned}
T_i & \equiv \sum_{\omega \in \Omega} q(\omega) c_i(\omega) + p \cdot N_i^+ - \bar{n}_i p \cdot \bar{N} - \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega) \, dN_{ij} \\
& = \sum_{\omega \in \Omega} \left[\alpha_i \frac{\partial U_i}{\partial c_i(\omega)} + \hat{\mu}_i(\omega) \right] c_i(\omega) + \hat{p} \cdot N_i^+ - \bar{n}_i \hat{p} \cdot \bar{N} - \sum_{\omega \in \Omega} \hat{q}(\omega) \int d_j(\omega) \, dN_{ij} \\
& = \sum_{\omega \in \Omega} \alpha_i \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) - \bar{n}_i \hat{p} \cdot N + \hat{p} \cdot N_i^+ - \int \hat{v}_{ij} dN_{ij}^+ \\
& = \sum_{\omega \in \Omega} \alpha_i \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) - \bar{n}_i \hat{p} \cdot \bar{N}
\end{aligned}
$$

where we substituted in the Planner's first-order and complementary-slackness conditions. Since aggregate ressource feasibility implies aggregate budget feasibility, it follows that $\sum_{i \in I} T_i = 0$. Since, in addition, $\sum_{i \in I} \bar{n}_i = 1$, we obtain that:

$$
\hat{p} \cdot N = \sum_{k \in I} \sum_{\omega \in \Omega} \alpha_k \frac{\partial U_k}{\partial c_k(\omega)} c_k(\omega).
$$

Since (c, N^+) satisfies the second condition stated in the Proposition, we obtain that $T_i = 0$, so budget balance holds.

E.6   Proof of Proposition C.6

Proof that $\Delta^\star(\alpha)$ is convex-valued. To show that $\Delta^\star(\alpha)$ is convex valued, we note that when $u_i(c)$ is strictly concave, $c_i(\omega)$ is uniquely determined, and so the term $\pi(\omega) u_i' [c_i(\omega)] c_i(\omega)$ is the same for all $(c, N^+) \in \Gamma^\star(\alpha)$. When $u_i(c)$ is linear, then $u'(c)c = c$ is linear. Taken together, this means that the function defining $\Delta^\star(\alpha)$ preserves the convexity of $\Gamma^\star(\alpha)$.

Proof that $\Delta^\star(\alpha)$ has a closed graph. Consider any converging sequence of α^k and $\Delta^k \in \Delta^\star(\alpha^k)$, generated by a sequence $(c^k, N^{k+}) \in \Gamma^\star(\alpha^k)$. Since $\Gamma^\star(\alpha^k)$ is included in the set of incentive feasible allocations, which by Lemma C.1 is weakly compact, we can extract a weakly convergent subsequence $(c^\ell, N^{\ell+})$ of (c^k, N^{k+}). Since we know from Proposition C.1 that $\Gamma^\star(\alpha)$ has a weakly closed graph, it follows that $\lim(c^\ell, N^{\ell+}) \in \Gamma^\star(\lim \alpha^\ell)$. If $u_i'(c)$ is continuously differentiable at $\lim c_i^\ell(\omega)$, then by continuity we have:

$$
\lim \left(\alpha_i^\ell u_i' \left[c_i^\ell(\omega) \right] c_i^\ell(\omega) \right) = \left(\lim \alpha_i^\ell \right) \times u_i' \left[\lim c_i^\ell(\omega) \right] \times \left(\lim c_i^\ell(\omega) \right).
$$

If $u_i(c)$ is not continuously differentiable at $\lim c_i^\ell(\omega)$ then given our maintained assumption that $u_i(c)$ is continuously differentiable over $(0, \infty)$, it must be that $\lim c_i^\ell(\omega) = 0$ and $u_i'(0) = -\infty$. Since $\lim c_i^\ell(\omega) = 0$ is part of a social

optimum, it must be that $\lim \alpha_i^\ell = 0$. But we know in this case from Proposition C.1 that

$$\lim \alpha_i^\ell u_i' \left[c_i^\ell(\omega) \right] c_i^\ell(\omega) = 0 = \lim \alpha_i^\ell u_i' \left[\lim c_i^\ell(\omega) \right] \lim c_i^\ell(\omega).$$

Taken together, we obtain that $\lim \Delta^\ell = \lim \Delta^k \in \Delta^\star(\lim \alpha^\ell) = \Delta^\star(\lim \alpha^k)$.

Proof that $\Delta^\star(\alpha)$ is bounded. Otherwise, there would exists some sequence α^k and $\Delta^k \in \Delta^\star(\alpha^k)$ such that $\max |\Delta_i^k| \to \infty$. Since α^k belongs to a compact set we can extract a converging subsequence α^ℓ. Since $\Delta^\star(\alpha)$ has a closed graph $\lim \Delta^\ell \in \Gamma^\star(\lim \alpha^\ell)$ and so must be finite, which is a contradiction.

An auxiliary fixed-point problem. Let M be such that $\max |\Delta_i| \leq M$ for all $\Delta \in \Delta^\star(\alpha)$ and $\alpha \in \mathcal{A}$. Let $\mathcal{D}$ be the set of transfers $\Delta = (\Delta_1, \ldots, \Delta_I)$ such that $\sum_{i \in I} \Delta_i = 0$ and $\max |\Delta_i| \leq M$. Finally, let $K(\alpha, \Delta)$ be the function $\mathcal{A} \times \mathcal{D} \to \mathcal{A}$ such that

$$K_i(\alpha, \Delta) = \frac{(\alpha_i - \Delta_i)^+}{\sum_{k \in I} (\alpha_k - \Delta_k)^+},$$

where x^+ denotes the positive part of x. For each $(\alpha, \Delta) \in \mathcal{A} \times \mathcal{D}$, let the set $\Phi(\alpha, \Delta)$ be the product of the singleton $\{K(\alpha, \Delta)\}$ and the set $\Delta^\star(\alpha)$. By construction, $\Phi(\alpha, \Delta) \subseteq \mathcal{A} \times \mathcal{D}$. Since $\sum_{k \in I} (\alpha_k - \Delta_k)^+ \geq \sum_{k \in I} (\alpha_k - \Delta_k) = 1 > 0$ it follows that $K_i(\alpha, \Delta)$ is a continuous function over $\mathcal{A} \times \mathcal{D}$. Given our earlier result that $\Delta^\star(\alpha)$ has a closed graph, this implies that the correspondence $\Phi(\alpha, \Delta)$ has a closed graph as well. This allows to apply Kakutani's fixed point Theorem (see Corollary 17.55 in Aliprantis and Border (1999)) and assert that Φ has a fixed point, i.e., there exists some $(\alpha, \Delta) \in \mathcal{A} \times \mathcal{D}$ such that

$$\begin{aligned} \alpha_i &= \frac{(\alpha_i - \Delta_i)^+}{\sum_{k \in I} (\alpha_k - \Delta_k)^+} \text{ for all } i \in I \\ \Delta &\in \Delta^\star(\alpha). \end{aligned}$$

Proof that all fixed-points are such that $\Delta_i = 0$ for all $i \in I$. Next, we show that a fixed point of Φ has the property that $\Delta_i = 0$ for all $i \in I$. Indeed if $\alpha_i = 0$, then from the definition of $\Delta^\star(\alpha)$ we have that $\Delta_i \leq 0$, and from the fixed-point equation that $(-\Delta_i)^+ = 0 \Leftrightarrow \Delta_i \geq 0$. Hence, if $\alpha_i = 0$, then $\Delta_i = 0$. If $\alpha_i > 0$, then from the fixed point equation

$$\alpha_i \times \sum_{k \in I} (\alpha_k - \Delta_k)^+ = \alpha_i - \Delta_i \Rightarrow \Delta_i = \alpha_i \times \left[1 - \sum_{k \in I} (\alpha_k - \Delta_k)^+ \right].$$

Hence, all Δ_i such that $\alpha_i > 0$ have the same sign. Since $\Delta_i = 0$ when $\alpha_i = 0$, it follows that all Δ_i have the same sign. But since $\sum_{i \in I} \Delta_i = 0$, this implies that $\Delta_i = 0$ for all $i \in I$.

E.7 Modified Security Market Line

Proposition E.1 *Suppose the distribution of tree supplies is strictly increasing. Let $R_j(\omega) = \frac{d_j(\omega)}{p_j}$ be the return of tree j, $R_m(\omega) = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell} R_j(\omega) d\bar{N}_j$ the market return, and $\beta_j = \frac{Cov(R_m, R_j)}{V(R_m)}$ the market beta of tree j. Then, β_j is a continuous and strictly decreasing function of j. Moreover, the expected return of tree j is a piecewise linear function of β_j:*

$$\mathbb{E}[R_j - R_f] = \beta_j \left(\mathbb{E}[R_m - R_f] - \theta_m \right) + \theta_j, \tag{61}$$

where

$$\theta_j = \theta_k - \phi \max(\beta_j - \beta_k, 0) - \psi \max(\beta_k - \beta_j, 0), \tag{62}$$

and $R_f = \left(\sum_{\omega \in \Omega} q(\omega)\right)^{-1}$ *is the risk-free rate,* $\theta_j = \Delta_j/p_j$, *is the (per dollar invested) discount induced by incentive constraints for tree* j, k *is the marginal tree,* $\phi > 0$, $\psi > 0$, *and* $\theta_m = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell} \theta_j d\bar{N}_j$ *is the average discount induced by incentive constraints. Equation (61) also holds for financial trees by setting* $\theta_j = 0$.

Proof that $j \mapsto \beta_j$ **is strictly decreasing.** Since there are only two states of nature, correlations are either equal to one, zero, or minus one. It follows from $R_m(\omega_1) < R_m(\omega_2)$ that $\beta_j = \frac{\sigma(R_j)}{\sigma(R_m)}\text{Sign}[d_j(\omega_2) - d_j(\omega_1)]$, where:

$$\left(\sigma(R_j)\right)^2 = \sum_{\omega \in \Omega} \pi(\omega)\left(\frac{d_j(\omega) - \bar{d}_j}{p_j}\right)^2 = \sum_{\omega \in \Omega} \pi(\omega)(1 - \pi(\omega))^2 \left(\frac{d_j(\omega_2) - d_j(\omega_1)}{p_j}\right)^2$$

Equation (11) implies that $p_j = a_i(\omega_1)d_j(\omega_1) + a_i(\omega_2)d_j(\omega_2)$, where i denotes the agent holding tree j and $a_i(\omega) > 0$. Thus:

$$\beta_j = \frac{1}{\sigma(R_m)} \left(\sum_{\omega \in \Omega} \pi(\omega)(1 - \pi(\omega))^2\right)^{\frac{1}{2}} \frac{\frac{d_j(\omega_2)}{d_j(\omega_1)} - 1}{a_i(\omega_1) + a_i(\omega_2)\frac{d_j(\omega_2)}{d_j(\omega_1)}}. \tag{63}$$

$\frac{d_j(\omega_2)}{d_j(\omega_1)} \mapsto \beta_j$ is clearly continuous away from the marginal tree k. And it is also continuous at the marginal tree since p_j is continuous at $j = k$. For $j \neq k$, we can take the derivative:

$$\frac{d\beta_j}{d\frac{d_j(\omega_2)}{d_j(\omega_1)}} = \frac{1}{\sigma(R_m)} \left(\sum_{\omega \in \Omega} \pi(\omega)(1 - \pi(\omega))^2\right)^{\frac{1}{2}} \frac{a_i(\omega_1) + a_i(\omega_2)}{\left(a_i(\omega_1) + a_i(\omega_2)\frac{d_j(\omega_2)}{d_j(\omega_1)}\right)^2} > 0.$$

Proof of equation (61). There is a different pricing kernel for each agent. For trees j held by agent i, the pricing kernel is:

$$1 = \mathbb{E}\left[\frac{q(\omega)}{\pi(\omega)}R_j(\omega)\right] - \delta\frac{\mu_i(\omega_i)}{\lambda_i}R_j(\omega_i).$$

Denoting the risk-free rate as $R_f = \left(E\left[\frac{q(\omega)}{\pi(\omega)}\right]\right)^{-1}$, the usual manipulations lead to:

$$\mathbb{E}[R_j(\omega) - R_f] = -R_f Cov\left(\frac{q(\omega)}{\pi(\omega)}, R_j(\omega)\right) + \theta_j,$$

where $\Delta_j = R_f \delta\frac{\mu_i(\omega_i)}{\lambda_i}R_j(\omega_i)$. Since there are two states of nature, $\frac{q(\omega)}{\pi(\omega)}$ can be written as an affine function of the market return with slope κ. Thus:

$$\mathbb{E}[R_j(\omega) - R_f] = -\kappa R_f Cov\left(R_m(\omega), R_j(\omega)\right) + \theta_j, \tag{64}$$

where $\theta_j = R_f \delta\frac{\mu_i(\omega_i)}{\lambda_i}R_j(\omega_i) = \frac{\Delta_j}{p_j}$. Multiplying by $\frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell}$ and integrating over j, we obtain the pricing kernel for the market portfolio:

$$\mathbb{E}[R_m(\omega) - R_f] = -\kappa R_f Var\left(R_m(\omega)\right) + \theta_m, \tag{65}$$

where $\Delta_m = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell} \theta_j d\bar{N}_j$. Combining (64) and (64) yields the modified CAPM formula (61).

Next, we show that θ_j can be written as a piecewise linear function of β_j with a kink at the marginal tree β_k. $R_j(\omega_1) = \frac{d_j(\omega_1)}{p_j} = \frac{1}{a_i(\omega_1) + a_i(\omega_2)b_j}$, where i denotes the agent holding tree j and $b_j \equiv \frac{d_j(\omega_2)}{d_j(\omega_1)}$. Equation (63) implies that β_j can be written as a function of b_j: $\beta_j = \rho_0 \frac{b_j - 1}{a_i(\omega_1) + a_i(\omega_2)b_j}$, where $\rho_0 = \frac{1}{\sigma(R_m)}\left(\sum_{\omega \in \Omega} \pi(\omega)(1 - \pi(\omega))^2\right)^{\frac{1}{2}}$. Inverting this function, we can write b_j as a function of β_j: $b_j = \frac{\rho_0 + \beta_j a_i(\omega_1)}{\rho_0 - \beta_j a_i(\omega_2)}$. Thus: $R_j(\omega_1) = \frac{\rho_0 - \beta_j a_i(\omega_2)}{(a_i(\omega_1) + a_i(\omega_2))\rho_0}$. Similarly: $R_j(\omega_2) = \frac{\rho_0 + \beta_j a_i(\omega_1)}{(a_i(\omega_1) + a_i(\omega_2))\rho_0}$. It implies that Δ_j is linear and decreasing in β_j for trees j held by agent 1 and linear and increasing for trees held by agent 2. It follows from the continuity of θ_j at the marginal tree k that

θ_j can be written as (62).