

# General Equilibrium Outside the Edgeworth Box: Debt, Fragility, and Multiplicity

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## Abstract

We extend General Equilibrium to include negative endowments, thus allowing for initial debt and also the possibility of default. We show, as in traditional GE, that equilibrium always exists, is Pareto efficient, and is locally unique. The new phenomenon is that no matter what the utilities, for roughly half the endowments, enough debt necessarily leads to fragile equilibria and then multiple equilibria. This paper establishes a robust link between debt, fragility, and multiplicity.

## 1 Introduction

At the height of debt crises, or just after they have passed, leaders often suggest that confidence can or did restore the economy. Franklin Roosevelt famously declared in his inaugural address in 1933 that “The only thing we have to fear is fear itself.” In the aftermath to the 2007-09 American debt crisis, U.S. Treasury Secretary Geithner (2015) and U.S. Federal Reserve Chairman Bernanke (2015) suggested that what had been needed to stem the crisis, was confidence, which “Stress Tests” and the “Courage to Act” provided. Similarly, Mario Draghi declared at the height of the European debt crisis in 2012 that “The ECB is ready to do whatever it takes to preserve the Euro. And believe me, it will be enough.” During the ongoing Greek debt crisis that continues to today, the government often said that it was crucial to restore confidence.

One interpretation of these accounts by government officials is that the speakers believed that in the middle of their debt crises there were multiple equilibria, and that by restoring confidence they could move the economy from a bad equilibrium to a better one. An alternative interpretation is that they felt their indebted economies were so fragile that the interventions of the government or central bank,

though small when measured on the scale of the entire American or European economies, could move entire economies from a terrible outcome to a normal outcome.

The question we ask in this paper is how or why does large debt lead to fragility or multiplicity of equilibria? An old answer to this question was suggested by Diamond and Dybvig (1983) for the special case of bank deposits. They described bank runs as multiple equilibria of games in which there is a common resource. Depositors have a joint claim on the assets of a bank, not individual claims to individual pieces of the assets. When such a common resource has long run value greater than total deposits, but liquidation value less than total deposits, there are two equilibria, one in which everybody leaves their money in, because each is sure the others will leave their money in, and another in which there is a run because each pulls her money out because she thinks the others are pulling their money out and there will be nothing left for those who wait.<sup>1</sup>

The global economy is not a big bank. Many authors nonetheless have drawn connections between the bank runs of the 1800s and early 1900s and the American crisis of 2008. Gorton (2010) for example likened the 2008 “collateral run” to the earlier bank runs. On the face of it, however, the analogy between bank runs, at least as described by Diamond and Dybvig (1983), and collateral runs seems imprecise. Unlike bank assets, each collateral is pledged to one lender. One lender’s withdrawal does not diminish the physical collateral held by another lender. There is no common pool of collateral. Thus it appears that there is a missing link in the collateral run story between debt and multiplicity.<sup>2</sup>

Another old story called the leverage cycle linked high leverage to the fragility of equilibrium.<sup>3</sup> In a highly leveraged economy, a small shock can lead to a big change in equilibrium asset prices and outcomes for two reasons. First, the most enthusiastic buyers of collateral are the ones who are most leveraged, and so a small fall in collateral value has a leveraged effect on their wealth, reducing their demand for the collateral. And second, when the bad news is coupled with a rise in volatility (scary bad news), the leverage ratios plummet so that new buyers can no longer borrow much to buy the asset, and its value falls still further. The leverage cycle thus connects debt to fragility through collateral and changes in volatility.

In this paper we wish to investigate the effect of debt itself on the fragility and multiplicity of perfectly competitive equilibrium, in which all resources are privately owned, without introducing banks or common resources, or collateral or volatility. Thus we analyze classical Walrasian exchange equilibrium with one twist: we allow for initial real debts. We make the novel observation that one can analyze any exchange

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<sup>1</sup>Since the long run value of the bank’s common assets exceeds the total debt to depositors, there is no loss to leaving the money in if the others are in for the long run, and since the total debt owed to depositors is more than the immediate liquidation value of the bank’s assets, each depositor rushes to get her money out first if she thinks the others are running to get theirs out.

<sup>2</sup>Gorton himself acknowledges that collateral runs are not analogous to bank runs ala Diamond and Dybvig (1983). In his view collateral runs and the old bank runs involved asymmetric information and adverse selection about what are truly safe assets.

<sup>3</sup>See for example Geanakoplos (1997, 2003, 2010) and Fostel-Geanakoplos (2008).

economy in which some agents have arbitrarily large negative endowments, which can be interpreted as (real) debts. In the special case with two goods and two agents, this is like thinking about endowments outside the Edgeworth Box. Nobody seems to have systematically analyzed such a model before, perhaps because it is not obvious how to deal with debts that cannot be paid.<sup>4</sup> We simply make the assumption that everyone is obliged to pay all she can, defaulting by the same percentage on each debt.<sup>5</sup>

We discover that debt magnifies income effects, provided that agents have different marginal propensities to consume. When debts are large enough, and go in the right direction, they create upward sloping demand, and multiple equilibria. Furthermore, on the way to becoming so large, debt flattens the slope of downward sloping aggregate demand. Thus an economy with big (but not huge) debts in the right direction will respond to small quantity perturbations with dramatic changes in price; the equilibrium is fragile. Walrasian economies with debt thus have three regimes. In the conventional regime, perturbations to endowments or utilities create small changes in equilibrium. In the fragile regime, small perturbations to endowments or utilities produce gigantic changes in equilibrium. In the multiple equilibria regime, equilibrium might jump because of a change in ‘confidence’, with no changes in endowments or utilities. Whether the American or Greek economy debts of 2007-12 fell into the fragile regime or the multiple equilibrium regime would need careful investigation. Our construction is thus a clarification, and not an endorsement or refutation, of officials’ claims that by restoring confidence they could move their economy to a better place.

Our formal model considers a Walrasian economy with an arbitrary number of commodities and agents and arbitrary real debts. Depending on market prices, agents may be forced to default. No rise in price levels can inflate the debts away, since they are denoted in real terms. We give a simple definition of Walrasian equilibrium with default, and prove that equilibria always exist, no matter how large the debts, and that generically, they are finite in number. Existence is somewhat surprising, because with arbitrary debts, agents might encounter situations where their income (after they default) is driven down to zero. Zero income is known occasionally to create discontinuities in demand. We prove nevertheless that for all strictly monotonic utilities and endowments, equilibrium must exist. Moreover, we show that for all utilities and endowments satisfying a weak heterogeneity hypothesis, there is an open set of debts (including debts of arbitrarily large size, but not necessarily of small size) that give rise to multiple

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<sup>4</sup>Balasko and Shell (1981) considered an economy with arbitrary nominal debts, but they only considered equilibria with no default. High enough price levels essentially inflate the debt away, making default unnecessary. Those economies typically have a continuum of equilibria with no default, one for each sufficiently high price level, because different price levels create different real debts, effectively reallocating the initial endowments between creditor and lender. In our economies with real debt there are generically only a finite number of equilibria, some of which may involve default. Cass and Pavlova (2004) did consider real endowments outside the Edgeworth Box, but only for Cobb-Douglas economies, and only for those measure zero endowment points where the budget lines tangent to Pareto optima cross. Again they did not consider default. We return to their model later.

<sup>5</sup>This is tantamount to saying that the initial endowments and receivables are all no-recourse collateral, so perhaps there is a sense in which collateral creeps into this model as well, but in a very simple way.

equilibria. The needed heterogeneity hypothesis is that there must be a point on the Pareto frontier at which two agents have different marginal propensities to consume some commodity. We also prove (under the weak heterogeneity hypothesis) that given any multiplier  $M > 0$ , there is an open set of debts at which some equilibrium price will change by at least  $M$  times the change in some endowment.

In standard Walrasian economies without debts, i.e. with endowments inside the Edgeworth Box, it is possible to have fragile equilibria or multiple equilibria. But with some popular and convenient utilities, like Cobb-Douglas utilities, multiplicity is impossible. Furthermore, there is no unified, thematic explanation for why multiplicity does occasionally arise in standard Walrasian economies. Here we show that for any utilities and endowments (with a trivial amount of heterogeneity) there must be a robust set of debts at which there are fragile equilibria and another robust set of debts, including debts of arbitrarily large size, at which there must be multiple equilibria. The fragility and multiplicity arise from the same thematic source, which is obscured in standard Walrasian economies that do not allow for negative endowments. When in equilibrium, agents that are selling a good have higher marginal propensities to buy the good than its buyers do, then income effects counteract the stabilizing substitution effects. Large debts create fragility and eventually multiplicity in two ways. First, when an agent has negative endowment of a good, she must be a buyer in equilibrium, even if she has low marginal propensity to consume the good. Second, the bigger the debt, the larger the net trades in equilibrium, and hence the larger the income effects, since income effects are the product of net trades and marginal propensities to consume. With really big debts, income effects must dominate substitution effects.

The intuition for multiplicity begins by supposing that an agent A has in the past incurred a very large debt denoted in a commodity (say food) in order to acquire another good (say houses). Starting from this situation, why should there be multiple equilibria? An incomplete explanation is that when the debt comes due, A is forced to sell houses in equilibrium to raise the food to pay off the debt. This is called a “fire” sale. The “forced” sale of houses is held to be responsible for the resulting low equilibrium price of houses. But in Walrasian equilibrium, every good is up for sale. One could just as easily say that A voluntarily chose to buy a large quantity of food because she has a high marginal utility of food when consumption is negative; much the same could be said if we substitute low quantity of food for negative quantity. And what has all this got to do with multiple equilibria and confidence? If a central authority simply announced a carefully chosen higher price of houses, how could that also clear all the markets?

The answer is that the potential multiplicity also depends on A having a higher marginal propensity to consume houses at the original equilibrium than B does, and the desire to purchase a sufficiently large quantity of food at the original equilibrium. These circumstances occasionally arise in standard Walrasian equilibrium. The role for large debt is simply to ensure the desire to buy enough food, so that the multiplicity must arise. This gives a rationale for perfectly competitive bank runs: if leverage

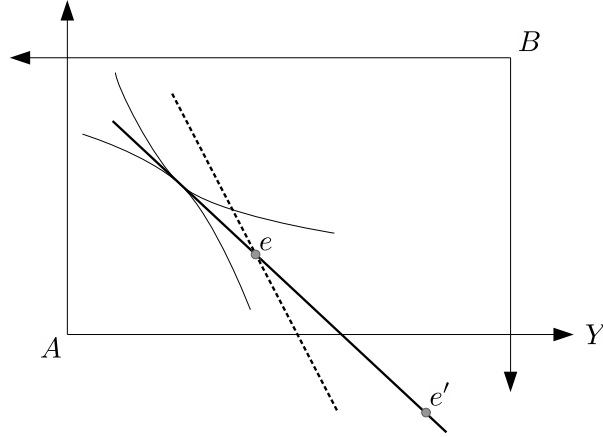


Figure 1: Endowments inside and outside the Edgeworth box can lead to the same equilibrium, but with different income effects.

is high enough, and in the right proportions, with debt used to finance net positions in the goods that the agents have a higher propensity to consume, then there must be multiple equilibria.

Consider the Edgeworth box diagram in Figure 1, where the endowment labeled  $e$  is in the Edgeworth Box and on the price line separating the two indifference curves at equilibrium. If the price of the good labeled  $Y$  rises, as indicated by the dotted line through  $e$ , the substitution effect leads both agents  $A$  and  $B$  to consume less of good  $Y$ . However, agent  $A$  gets richer, and if  $A$  has positive marginal propensity to consume  $Y$  out of income, then his income effect counteracts his substitution effect. On the other hand, the reverse is true of  $B$ , and his income effect reinforces his substitution effect. Very likely the substitution effects, which all go in the same direction, dominate.

The essential point is that the income effect of each agent is the product of his marginal propensity to consume at the equilibrium with the size of his excess demand (that is the difference between  $e$  and equilibrium consumption). The excess demand of agent  $A$  is the negative of the excess demand of agent  $B$ . Moving the endowment along the budget line proportionally increases both agents' excess demands without changing their marginal propensities to consume at the unchanged equilibrium. Suppose that  $A$  has a larger marginal propensity to consume  $Y$  than  $B$  does at the equilibrium. Then on net the aggregate income effects tend to counteract the substitution effects. The further away from the equilibrium we move the endowment along the equilibrium price line, the larger the income effects become, and on net, the more they counteract the substitution effect. The original aggregate excess demand at endowment  $e$ , given by the downward sloping line, becomes flatter and flatter, becoming very flat at endowment  $e'$ . From this initial endowment point, that a small shock in supply, or a small transfer of income from  $A$  to  $B$ , will cause a very major change in the equilibrium price of houses. The equilibrium has become fragile. This is one of the forces at work in the leverage cycle. But it does not require multiplicity of equilibria.

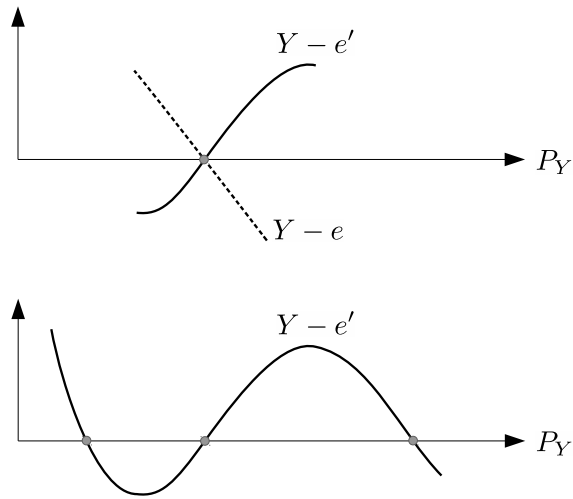


Figure 2: Excess demand for Y becomes upward sloping in response to increased leverage along the budget line, which leads to multiple equilibria.

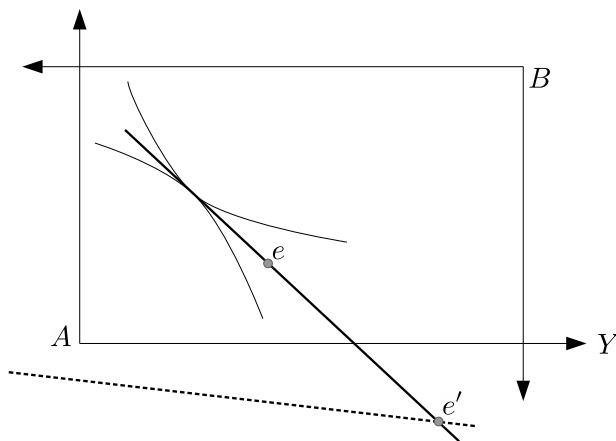


Figure 3: At some prices default is inevitable, and must be defined.

As the endowment moves still further away from the equilibrium consumption allocation, until it eventually slopes upward, as in the solid line depicting aggregate excess demand at  $e'$ . Eventually, perhaps only after the endowment leaves the Edgeworth Box, as at point  $e'$  in Figure 1, the income effect will dominate, and supply will become upward sloping around the equilibrium price, as seen in the upper panel of Figure 2.

As the lower panel of Figure 2 demonstrates, upward sloping demand at an equilibrium price guarantees the existence of two more equilibria, provided that we know that excess demand is always positive when price is near 0, and negative when the price is near infinity.

The interesting point is that in order for the income effect to dominate, we might need to put the initial endowment outside the Edgeworth Box. And then how do we know demand is well defined at all prices? Consider for example Figure 3. At the dotted price line, agent A cannot achieve any non-negative consumption. He must default.

Thus we must define default and repayment, and individual and aggregate demand, for any specified prices and endowments/debts. Once we do, we shall show that no matter what the endowments, and prices, demand is well defined and continuous. Moreover as price tends to zero, demand tends to infinity, and as price tends to infinity, excess demand turns negative. This guarantees the existence of multiple equilibria as shown in the diagrams. Moreover, with the use of the Borsuk-Ulam Theorem, this proof for two agents and two goods can be extended to an arbitrary number of goods and agents.

We emphasize again that debt by itself does not create upward sloping demand and multiplicity. The debt must be sufficiently large. A simpler and more general phenomenon arises from debt when demand is still downward sloping. When A has higher marginal propensity than B to consume housing, then more debt from A tends to counteract the substitution effect, flattening the aggregate excess demand for housing. A failure of confidence and a leverage crash can both create very low prices of housing. The former requires a huge amount of debt, and multiple equilibria. The latter requires less debt, and a small shock, and is consistent with a unique equilibrium.

In sections 2-4 we define Walrasian economies with debt, and show that for any prices there are uniquely defined repayment rates  $r_h$ , at which every agent  $h$  repays all of his debts, such that every agent is paying back as much debt as he can given the rates at which others repay him. This shows that demand is a well-defined function of prices alone. In section 5 we define Walrasian equilibrium with debts and in section 6 show that it always exists. In section 7 we describe the multiplicity theorem. In section 8 we prove that generically there will be a finite number of equilibria, so that the multiplicity we describe does not lead to a continuum of equilibria. In Section 9 we give a pictorial example of three equilibria in the Edgeworth Box.

In section 10 we return to the question of the debt repayment rate  $r_h$  for each agent  $h$ , and how it

depends on the repayment rates of  $r_i \neq r_h$ . We observe that the uniqueness of consistent delivery rates stems from the fact that no resources are lost to the economy in case of default. When resources are lost, there can be a lattice of self consistent delivery rates, and thus additional multiplicity. We call this default multiplicity, in contrast to the earlier leverage multiplicity.

There are at least two precursors to our work. Cass and Pavlova (2004) consider Cobb-Douglas economies with heterogeneous consumers, and they show that the separating hyperplanes at each Pareto optimal allocation are bound to intersect somewhere outside the Edgeworth Box, i.e. somewhere with negative endowments. At these intersection points, there will be multiple equilibria, possibly of many dimensions. Cass and Pavlova (2004) do not define demand or equilibrium at any other endowment points, and they do not consider default. As Cass and Pavlova point out, the set of intersection points that give rise to multiplicity has measure zero, and thus their multiplicity is completely non-robust. We show that there is a robust, open set of endowment points that give rise to multiple equilibria. Moreover, generically there will be a finite number of equilibria. Hence even in the Cobb-Douglas case, our endowments with multiple equilibria are disjoint from those in Cass and Pavlova (2004).

Brunnermeier and Pedersen (2009) give an example of a 3 period economy in which multiple equilibria arise in one state in the second period, starting from the endowments inherited from the first period in which debt was incurred to buy securities. Neither Cass and Pavlova (2004) nor Brunnermeier and Pedersen (2009) mention the role increased debt plays in magnifying income effects that can reverse substitution effects and create upward sloping demand, provided that agents display different marginal propensities to consume. And both of those papers work with special, parameterized examples.

## 2 Economy

The consumption space is  $\mathbb{R}_+^L$  and the set of agents is  $H = \{1, \dots, H\}$ . Agents have utilities

$$u^h : \mathbb{R}_+^L \rightarrow \mathbb{R}$$

that are continuous, concave, and strictly monotonic.

The innovation is that the endowments of each agent  $h \in H$  are given not only by the usual

$$e^h \in \mathbb{R}_{++}^L$$

but also by additional vectors

$$d^{hi} \in \mathbb{R}_+^L, i \neq h$$

where  $d_\ell^{hi} \geq 0$  denotes the amount of good  $\ell$  that agent  $h$  owes agent  $i$ .



### 3 Budget Set

We assume that every agent  $h$  either delivers completely or else defaults by the same percentage on every debt. Let  $r_i \in [0, 1]$  be the rate of delivery by agent  $i$ , that is, the fraction of the debt owed by  $i$  that  $i$  actually delivers. Let  $r = (r_1, \dots, r_H) \in [0, 1]^H$ . The income of any agent  $h \in H$  is then

$$I^h(p, r) = p \cdot e^h + \sum_{i \neq h} r_i p \cdot d^{ih} - r_h \sum_{i \neq h} p \cdot d^{hi}$$

The budget set of agent  $h$  is

$$B^h(p, r) = \{x \in R_+^L : p \cdot x \leq I^h(p, r)\}$$

We suppose that  $r_h$  is determined by collateral owned by agent  $h$ , which simultaneously backs all her debts. Two definitions are presented, depending on whether physical goods are the only collateral, or whether debts can collateralize debts. Note that the latter pyramiding collateral involves an infinite regress, since how much  $i$  defaults affects how much  $j$  defaults which affects how much  $i$  defaults, etc. So we start with the simpler definition. But we shall focus on the more interesting pyramiding collateral case.

#### 3.1 Budget Set without Pyramiding

We suppose that if an agent defaults, her endowments  $e^h$  can be seized, but not the income she receives from others' debts to her. Note that income from debt receivables will be usable by  $h$  for consumption no matter what  $h$ 's own debts are. They are not collateralized by  $h$ 's debts. If  $p \cdot e^h \geq p \cdot \sum_{i \neq h} d^{hi}$ , then the agent must fully deliver and  $r_h = 1$ . If  $p \cdot e^h < p \cdot \sum_{i \neq h} d^{hi}$ , then all her collateral will be seized and she will default. Her delivery rate in both cases can be succinctly written as

$$r_h = f_h(p) = \begin{cases} 1 & p \cdot \sum_{i \neq h} d^{hi} = 0 \\ \min\left(1, \frac{p \cdot e^h}{p \cdot \sum_{i \neq h} d^{hi}}\right) & p \cdot \sum_{i \neq h} d^{hi} > 0 \end{cases}$$

Notice that  $r_h$  does not depend on  $r_i$ .

#### 3.2 Budget Set with Pyramiding

Alternatively we can imagine that  $h$ 's receivables  $\sum_{i \neq h} r_i p \cdot d^{ih}$  are also collateral for her debts. Then we get

$$r_h = g_h(p, r) = \begin{cases} 1 & p \cdot \sum_{i \neq h} d^{hi} = 0 \\ \min\left(1, \frac{p \cdot e^h + \sum_{i \neq h} r_i p \cdot d^{ih}}{p \cdot \sum_{i \neq h} d^{hi}}\right) & p \cdot \sum_{i \neq h} d^{hi} > 0 \end{cases}$$

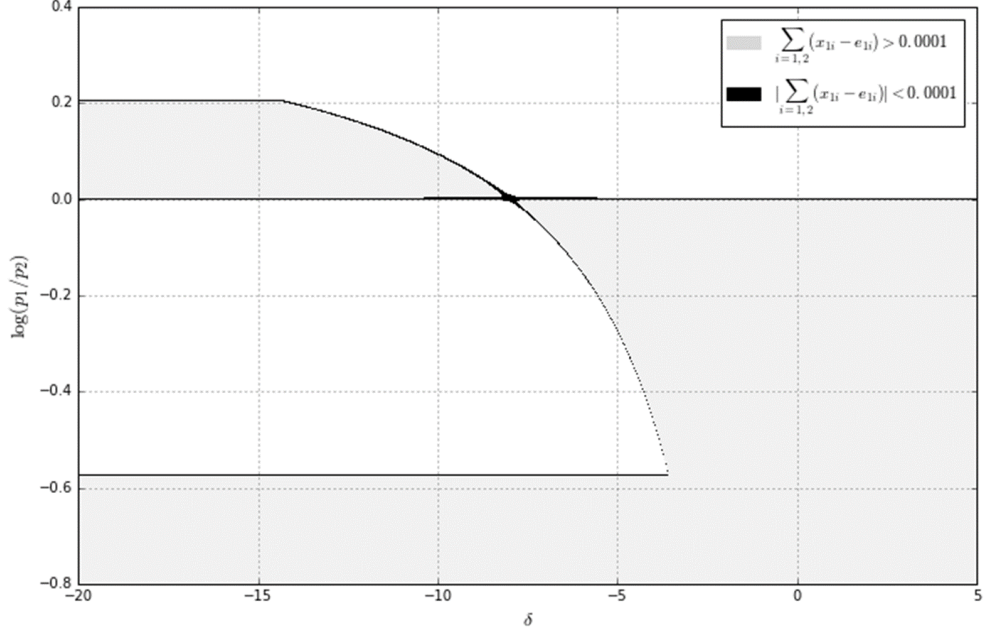


Figure 4: Equilibrium correspondence for the  $\delta$  indexed economy of Section 4.

This time  $r_h$  does depend on  $r_i$ . We call this pyramiding collateral because  $i$ 's debt to  $h$  is being used as collateral for  $h$ 's debt to  $k$ , which in turn is being used as collateral for  $k$ 's debt to  $m$ .

## 4 Example

For a scalar  $\delta$  consider the following two good two agent Walrasian endowment economy with debt

$$u_i = \left( (\alpha_i)^{1/s_i} (x^{1i})^{(s_i-1)/s_i} + (1 - \alpha_i)^{1/s_i} (x^{2i})^{(s_i-1)/s_i} \right)^{s_i/(s_i-1)}$$

$$\alpha_1 = 1/2, \alpha_2 = 1/3, s_1 = 2, s_2 = 1/2,$$

$$e^1 = (1, 1), e^2 = (1, 2),$$

$$d^{12} = (\max(0, \delta), \max(0, -\delta)), d^{21} = (\max(0, -\delta), \max(0, \delta)).$$

Regardless of  $\delta$ ,  $p_1 = p_2$  is always an equilibrium of this economy. Indeed from an accounting perspective such debt are of no consequence. But  $p_1 = p_2$  is not always the only equilibrium. Figure 4 presents the equilibrium correspondence with  $\delta$ . We see that for  $\delta$  values that are low enough there are two additional equilibria.

To see that multiplicity is assured for high enough leverage, and a geometric version of the main proof for the two good two agent case, it is helpful to examine the Edgeworth box presented in Figure

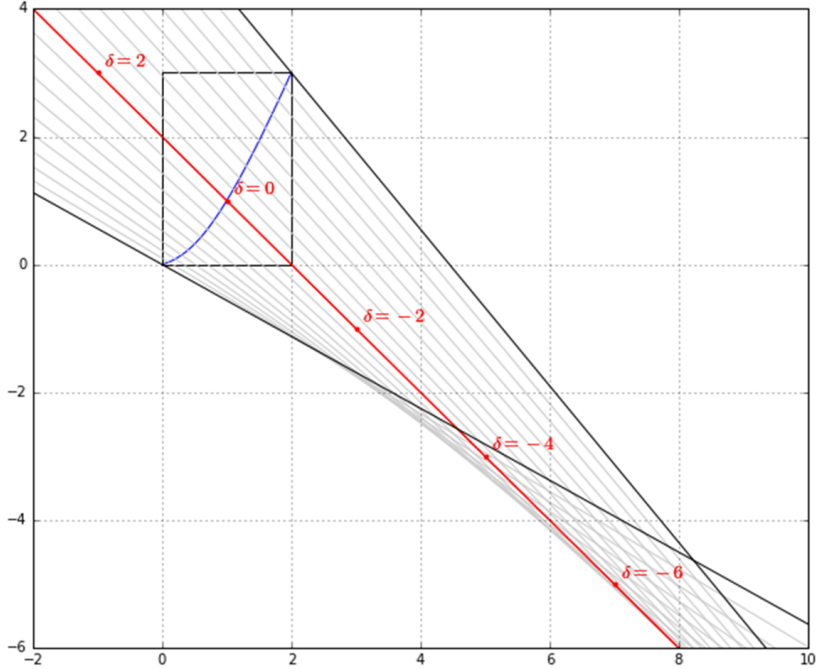


Figure 5: Edgeworth box for the  $\delta$  indexed economy of Section 4.

5. Because debts allow the effective endowment to lie outside the Edgeworth box, Figure 5 extends the view to include the surroundings of the Edgeworth box. The blue line describes the set of Pareto optimal allocations. The red line describes the budget line in the  $p_1 = p_2$  equilibrium. Because changing  $\delta$  changes debt in a way that does not transfer wealth in the  $p_1 = p_2$  equilibrium, it amounts to shifting the effective endowment along the red line. Each gray line presents the budget line for another equilibrium, and the black lines present the budget line in the any equilibria that involve zero consumption for one of the agents.

As long as the red line does not intersect another line, the original equilibrium remains unique. But because the other budget lines have different slopes, there must be a points where the red line meets each of the other budget lines. Each meeting point is an economy that sustains multiple equilibria, one for each meeting budget line. Moreover, any point on the red line that is beyond a black line also supports the associated default equilibrium. The example demonstrates why generally there must be an open set of economies that generate multiple equilibria because it shows that all that is needed is different sloped budget lines.

The reason for the appearance of multiplicity is the intensification of wealth effects generated by

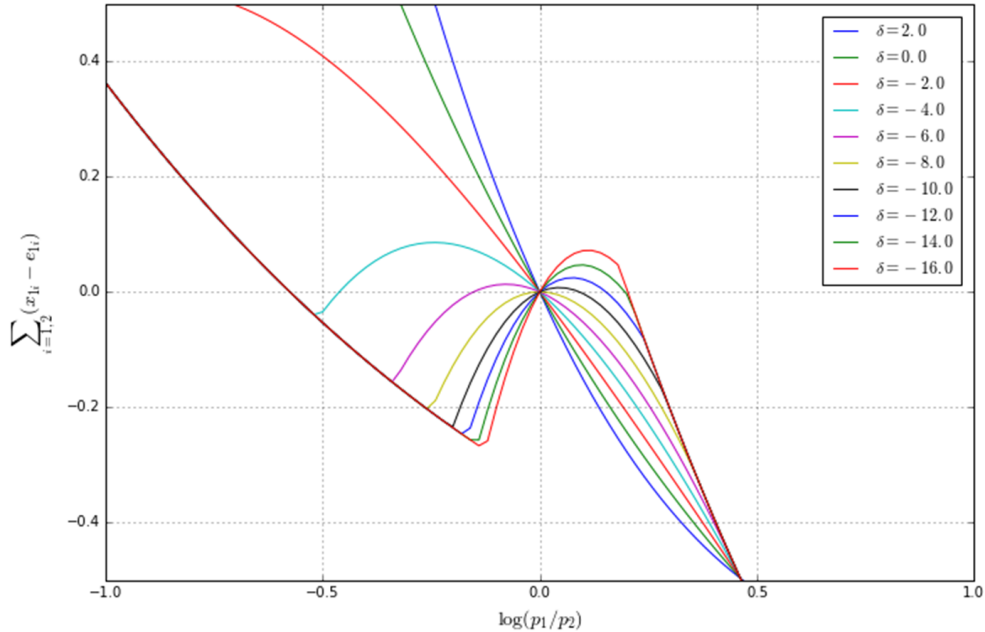


Figure 6: Excess demand for good 1 for the  $\delta$  indexed economy of Section 4.

higher debts. By the Slutsky equation

$$\frac{dx^h}{dp} = K^h + v^h \left( x^h - p \cdot e^h - \sum_{i \neq h} r_i p \cdot d^{ih} + r_h \sum_{i \neq h} p \cdot d^{hi} \right)^T \quad h \in \{1, 2\}$$

where  $K^h$  is a matrix of substitution effects and  $v^h$  is a vector of wealth effects. When  $v^h$  differs across agents, changing  $\delta$  alters the slope of excess demand around the equilibrium, as presented in Figure 6. Increasing the slope of excess demand moves us always closer to multiplicity. Indeed, when the slope turns positive multiplicity becomes assured for any continuous demand function because the demand function is required to cross zero at least two additional times with a descending slope. This is the idea of the proof for the multiple good multiple agent case of Section 9, with the determinant of the Jacobian of excess demand generalizing the notion of slope.

## 5 Uniqueness of Consistent Delivery Rates

From now on we concentrate on pyramiding collateral. Our first goal is to show that the self-referential default rates are actually always uniquely determined.

**Theorem 1.** *Fix  $p \in R_{++}^L$ . Then there exists a unique  $r \in [0, 1]^H$  such that  $r = g(p, r)$ . Moreover, writing the solution  $r = r^*(p)$ , the function  $r^*(p)$  is continuous.*

The function  $g_p : [0, 1]^H \rightarrow [0, 1]^H$  defined by  $g_p(r) = g(p, r)$  is continuous in  $r$ , since each coordinate  $g_h(p, r)$  is continuous in both variables, and weakly vector increasing, in the sense that  $s \geq r \Rightarrow g_p(s) \geq g_p(r)$ , since each coordinate is weakly increasing. Our first step is to prove the existence of a consistent  $r$ .

**Lemma 2.** *Fix  $p \in R_{++}^L$ . If  $a \in [0, 1]^H$ , and  $g_p(a) \geq a$ , then there exists  $b \in [0, 1]^H$  with  $g_p(b) = b \geq a$ . In particular, there exists at least one fixed point  $r \in [0, 1]^H$  with  $g_p(r) = r$ .*

*Proof.* If  $g_p(a) \geq a$ , then by the monotonicity of  $g_p$ ,  $g_p^2(a) \geq g_p(a)$ , and more generally,  $g_p^n(a)$  is increasing in  $n$ . By the monotonicity and continuity of  $g_p$ , and the compactness of  $[0, 1]^H$ ,  $g_p(b) = b = \sup_n g_p^n(a) \geq a$ . Taking  $a = 0$  shows that there is indeed at least one fixed point  $r \geq 0$ .  $\square$

Our next step is to prove uniqueness. Along the way we shall derive some interesting properties of consistent default rates, including the fact that at least one agent fully delivers.

**Lemma 3.** *If  $g_p$  has two distinct fixed points, then it has distinct fixed points  $b \geq a$ .*

*Proof.* Let  $g_p(a) = a \neq b = g_p(b)$ . Then denoting the component-wise maximum by  $c = a \vee b$ ,  $g_p(c) \geq g_p(a) = a$  and  $g_p(c) \geq g_p(b) = b$ , hence  $g_p(c) \geq c = a \vee b$ . By Lemma 2, there is a fixed point  $g_p(r) = r \geq a \vee b$ .  $\square$

The next lemma shows that whatever is delivered goes to somebody else, so total income is always  $p \cdot \sum_{h \in H} e^h$ .

**Lemma 4.** *For all  $(p, r) \in R_{++}^L \times [0, 1]^H$ ,*

$$\sum_{h \in H} I^h(p, r) = p \cdot \sum_{h \in H} e^h$$

*Proof.* Obvious by plugging in the definition of  $I^h(p, r)$ .  $\square$

**Corollary 5.** *Fix  $p \in R_{++}^L$ . Suppose  $r = \bar{g}(p, r)$ . Then some  $r_h = 1$ .*

*Proof.* If  $r_h < 1$ , then  $I^h(p, r) = 0$ . If  $r_h < 1$  for all  $h \in H$ , then  $\sum_{h \in H} I^h(p, r) = 0 < p \cdot \sum_{h \in H} e^h$ , contradicting Lemma 4.  $\square$

**Lemma 6.** *Suppose  $H_0 \subset H$  and for all  $h \in H_0, i \in H \setminus H_0, d^{hi} = 0$ . Then for all  $(p, r) \in R_{++}^L \times [0, 1]^H$ ,*

$$\sum_{h \in H_0} I^h(p, r) \geq p \cdot \sum_{h \in H_0} e^h$$

*Proof.* Again, by the definition of  $I^h(p, r)$ .  $\square$

**Lemma 7.** *Suppose  $r = g(p, r)$ . Then if  $s \geq r$ , and  $s = g(p, s)$ , then  $r = s$ .*

*Proof.* Let  $H_0 = \{h \in H : r_h < s_h\}$ . If  $h \in H_0$ , then  $r_h < s_h \leq 1$ , and so  $I^h(p, r) = 0$ ; hence  $I^h(p, s) \geq 0 = I^h(p, r)$ . For  $i \in H \setminus H_0$ , which must exist due to Corollary 5, we also have  $I^i(p, s) \geq I^i(p, r)$ , because  $I^i$  is increasing in  $r_{-i}$  and  $r_i = s_i$ . By Lemma 6, we cannot have  $d^{hi} = 0$  for all  $h \in H_0, i \in H \setminus H_0$ , because that would imply  $0 = \sum_{h \in H_0} I^h(p, r) \geq p \cdot \sum_{h \in H_0} e^h > 0$ . So for some  $h \in H_0, i \in H \setminus H_0$  we have  $p \cdot d_i^{hi} > 0$ . But so  $s_h p \cdot d^{hi} > r_h p \cdot d^{hi}$  and hence  $I^i(p, s) > I^i(p, r)$ . Thus  $\sum_{i \in H} I^i(p, s) > \sum_{i \in H} I^i(p, r)$ , contradicting Lemma 4.  $\square$

Now to the main:

Proof of Theorem 1: By Lemma 3 if  $g_p$  has two distinct fixed points, then it has distinct fixed points  $b \geq a$ . But by Lemma 7 two fixed points  $b \geq a$  cannot be distinct. We therefore conclude that  $g_p$  cannot have two distinct fixed points. Thus the correspondence  $r^*(p)$  is actually a function. Furthermore, since the correspondence is defined by a continuous function  $g$ , it is clearly upper semi continuous. But any upper semi continuous function is continuous. QED

## 6 Walrasian Equilibrium with Debt, Default, and Pyramiding

Given a Walrasian economy with debts  $(u^h, e^h, d^{hi})_{h,i \in H}$ , define a Walrasian equilibrium with pyramiding by  $(p, (x^h, r_h)_{h \in H})$  satisfying

1.  $\sum_{h \in H} x^h = \sum_{h \in H} e^h$  and for all  $h \in H$ ,
2.  $x^h \in B^h(p, r)$
3.  $x^h \in B^h(p, r) \Rightarrow u^h(x^h) \geq u^h(x)$
4.  $r_h = g_h(p, r)$

## 7 Existence

**Theorem 8.** *Every Walrasian economy with debts  $(u^h, e^h, d^{hi})_{h,i \in H}$  has a Walrasian equilibrium with pyramiding  $(p, (x^h, r_h)_{h \in H})$ .*

Our proof is similar to the standard Walrasian existence proof. We begin by showing that the budget set is a continuous correspondence. For all  $(p, r) \in R_{++}^L \times [0, 1]^H$ , define  $I_+^h(p, r) = \max(0, I^h(p, r))$  and  $B_+^h(p, r) = \{x \in R_+^L : p \cdot x \leq I_+^h(p, r)\}$ .

**Lemma 9.**  $B_+^h(p, r)$  is a nonempty, convex-valued, continuous correspondence at all  $(p, r) \in R_{++}^L \times [0, 1]^H$ .

*Proof.* Clearly  $0 \in B_+^h(p, r)$  and clearly  $B_+^h(p, r)$  is convex. The correspondence is defined by continuous inequalities, so it is upper semi-continuous. To show lower semi-continuity, let  $(p, r) \in R_{++}^L \times [0, 1]^H$  and  $x \in B_+^h(p, r)$  be given. Suppose  $(p(n), r(n)) \rightarrow_n (p, r)$ . We must find  $x(n) \in B_+^h(p(n), r(n))$  with  $(x(n)) \rightarrow_n x$ . We consider two cases. If  $x = 0$ , take  $x(n) = 0$  for all  $n$ . If  $x \neq 0$ , then  $I^h(p, r) = p \cdot x > 0$ . By continuity  $I^h(p(n), r(n)) \rightarrow I^h(p, r)$ , and  $p(n) \cdot x \rightarrow_n p \cdot x > 0$ . In that case let  $x(n) = x \frac{\min(I^h(p(n), r(n)), p \cdot x)}{p(n) \cdot x}$ .  $\square$

**Lemma 10.** The budget sets  $\beta^h(p) = B^h(p, r^*(p))$  are continuous correspondences on  $R_{++}^L$ .

*Proof.* The function  $r^*(p)$  is continuous, and over its range,  $I^h(p, r) \geq 0$ . Hence, by Lemma 9,  $\beta^h(p) = B^h(p, r^*(p)) = B_+^h(p, r^*(p))$  is a continuous correspondence.  $\square$

**Lemma 11.** The demand correspondences  $\chi^h$  defined by

$$\chi^h(p) = \arg \max_{x \in \beta^h(p)} [u^h(x)]$$

are nonempty and convex valued and upper semi-continuous on  $R_{++}^L$ .

*Proof.* This follows from the maximum principle and the continuity of the budget correspondences  $\beta^h$ , established in Lemma 10.  $\square$

**Lemma 12.** The aggregate excess demand correspondence  $z(p) = \sum_{h \in H} (\chi^h(p) - e^h)$  is non-empty and convex valued and USC, and satisfies Walras Law (even though individual excess demand does not).

*Proof.* Lemma 4 guarantees Walras Law, and Lemma 11 gives USC.  $\square$

**Lemma 13.** For  $p \in \Delta_{++}^{L-1}$ , as some price  $p_\ell \rightarrow 0$ , aggregate excess demand for some good must go to infinity.

*Proof.* By Lemma 4 some agent's wealth must stay bounded away from 0, and by strict monotonicity his demand goes to infinity. The other agents' demands are bounded below by 0.  $\square$

Hence can apply standard argument. QED

## 8 Generic Finiteness of Equilibria

Our goal is to show that Walrasian equilibria with enough debts almost always have multiple equilibria. But we begin by showing that the multiplicity is still finite. Our theorem shows that generically, equilibria are finite in number.

We restrict attention to smooth preferences, that is utilities  $u^h$  that are infinitely differentiable, that satisfy  $Du^h(x) \gg 0$ ,  $z'D^2u^h(x)z > 0$ ,  $\forall x \in R_+^L$ ,  $z \in R^L/\{0\}$ .

**Theorem 14.** *Fix smooth  $(u^h)_{h \in H}$ , and  $(d^h)_{h,i \in H} \in R_+^{H(H-1)L}$ . Then for almost all  $(e^h)_{h \in H} \in R_{++}^{HL}$ , the economy  $(u^h, e^h, d^h)_{h,i \in H}$  has a finite number of equilibria  $(p, (x^h, r_h)_{h \in H})$ .*

Proof of Theorem 14:

First we consider all economies  $(u^h, e^h, d^h)_{h,i \in H}$  and equilibria  $(p, (x^h, r_h)_{h \in H})$  in which the agents can be partitioned into two disjoint subsets  $H_0$  in which  $r_h < 1$  and  $H_1$  for which  $I^h > 0$ .

To that end, let us fix a partition of  $H$  into disjoint subsets  $H_0$ , which will include only agents who default, and  $H_1 \neq \emptyset$ , which will include only agents who fully deliver. For variables  $p = (p_1, \dots, p_{L-1}, 1) \in R_{++}^{L-1}$ ,  $I = (I^h)_{h \in H_1} \in R_{++}^{H_1}$ ,  $r = (r_h)_{h \in H_0} \in (0, 1)^{H_0}$  and  $h \in H_1$ , let us define

$$\chi^h(p, I^h) = \operatorname{argmax}_{x \in \{x \in R_+^L : p \cdot x \leq I^h\}} [u^h(x)]$$

By the smoothness hypothesis on preferences,  $\chi^h$  is a smooth function. So is

$$\chi(p, I) = \sum_{h \in H_1} \chi^h(p, I^h)$$

Let  $\hat{\chi}(p, I)$  be the first  $L-1$  coordinates of  $\chi(p, I)$ . Similarly, let  $\hat{e}$  be the first  $L-1$  coordinates of  $e = \sum_{h \in H} e^h$ . Let  $\hat{z}(p, I, r, (e^h)_{h \in H}) = \hat{\chi}(p, I) - \hat{e}$ .

Similarly, for  $h \in H_1$ , let us define

$$\eta_h(p, I, r, (e^h)_{h \in H}) = I^h - [p \cdot e^h - \sum_{i \neq h} p \cdot d^{hi} + \sum_{i \in H_1} p \cdot d^{ih} + \sum_{i \in H_0} r_i p \cdot d^{ih}]$$

Finally, for  $h \in H_0$ , let us define

$$\rho_h(p, I, r, (e^h)_{h \in H}) = r_h \sum_{i \neq h} p \cdot d^{hi} - [p \cdot e^h + \sum_{i \in H_1} p \cdot d^{ih} + \sum_{i \in H_0} r_i p \cdot d^{ih}]$$

The map

$$F : R_{++}^{L-1} \times R_{++}^{H_1} \times (0, 1)^{H_0} \times R_{++}^{H_1 L} \rightarrow R^{L-1} \times R^{H_1} \times R^{H_0}$$



defined by

$$F(p, I, r, (e^h)_{h \in H}) = (\hat{z}(p, I, r, (e^h)_{h \in H}), (\eta_h(p, I, r, (e^h)_{h \in H}))_{h \in H_1}, (\rho_h(p, I, r, (e^h)_{h \in H}))_{h \in H_0})$$

is smooth. Moreover, we claim that  $F \neq 0$ . To see why, let us show that  $DF$  has full rank whenever  $F = 0$ .

Take  $h \in H_1$ . Observe first that by decreasing  $e_L^h$  by  $\varepsilon$  while increasing  $e_L^h$  by  $p_1\varepsilon$ , we leave  $p \cdot e^h$  and thus  $\chi$  unchanged, and so increase  $\hat{z}_\ell$  by  $\varepsilon$  without affecting any other coordinate of  $F$ . Thus  $\frac{d\hat{z}}{de^h}$  has full rank, and no other coordinate is affected by these changes in  $e^h$ .

Observe next that by decreasing  $e_L^h$  by  $\varepsilon$  we increase  $\eta_h$  by  $\varepsilon$  without affecting any other coordinate of  $F$ . Thus  $\frac{d\eta}{(de_L^h)_{h \in H_1}}$  has full rank, and no other coordinate is affected by these changes in  $e_L^h$ .

Take  $h \in H_0$ . By decreasing  $e_L^h$  by  $\varepsilon$  we increase  $\eta_h$  by  $\varepsilon$  without affecting any other coordinate of  $F$ . Thus  $\frac{d\rho}{(de_L^h)_{h \in H_1}}$  has full rank, and no other coordinate is affected by these changes in  $e_L^h$ .

By the transversality theorem, for almost all  $(e^h)_{h \in H}$ , the equations  $F(p, I, r, (e^h)_{h \in H}) = 0$  have a finite number of equilibria. Since there are only a finite number of subsets  $H_1$  of the finite set  $H$ , there can be only a finite number of equilibria. The only other equilibria involve agents  $h$  who at equilibrium pay all their debts in full but are left with exactly zero income. A nearly identical transversality argument shows that for almost all  $(e^h)_{h \in H}$ , this cannot happen. QED.

## 9 Leverage Multiplicity

**Theorem 15.** *Consider any standard Walrasian exchange economy with fixed endowments and smooth utilities  $(u^h, e^h)_{h \in H}$ , with standard Walrasian equilibrium  $(p, (x^h)_{h \in H})$ . Suppose that at this equilibrium the Marginal Propensities to Consume from wealth are not all identical across agents. Then there is an open set of debts  $(d^{hi})_{h, i \in H}$  (including points with arbitrarily large debt) such that each Walrasian economy with debts  $(u^h, e^h, d^{hi})_{h, i \in H}$  has multiple equilibria, including at least one with no default that is close to the original equilibrium.*

Proof of Theorem 15:

For any debts  $(d^{hi})_{h, i \in H}$ , define the extra endowment each  $h$  would get if there were no defaults by  $\delta^h = \sum_{i \neq h} d^{ih} - \sum_{i \neq h} d^{hi}$ . If  $p \cdot \delta^h = 0$  for all  $h$ , the same equilibrium with no defaults will prevail (and agents will all be spending strictly positive amounts on consumption). By continuity, for all nearby prices and debts nearby  $(d^{hi})_{h, i \in H}$ , demand will also involve no default. Hence demand is also smooth around the old equilibrium price vector  $p$  and debts  $(d^{hi})_{h, i \in H}$ .

By Slutsky's Theorem (see Barten et al. (1969)) we can write the derivative of each agent's demand

at  $p$  by

$$\frac{dx^h}{dp} = Dx^h = K^h + v^h \left( (e^h - x^h) + \delta^h \right)^T$$

where (i)  $K^h$  is symmetric and negative semidefinite, (ii)  $\text{rank}(K^h) = L - 1$  (iii)  $p^T K^h = K^h p = 0$ , (iv)  $p \cdot v^h = 1$ .

From now on, by adding a hat we drop the last row and column from each matrix. (We are normalizing the last price  $p_L = 1$ , and ignoring demand for the last good, giving us a “truncated” demand.) Denote the derivative of aggregate demand by

$$D\hat{x} = \sum_{h \in H} D\hat{x}^h = \sum_{h \in H} \hat{K}^h + \sum_{h \in H} \hat{v}^h \left( (\hat{e}^h - \hat{x}^h) + \hat{\delta}^h \right)^T = \hat{K} + \sum_{h \in H} \hat{v}^h \left( (\hat{e}^h - \hat{x}^h) + \hat{\delta}^h \right)^T$$

Note that after dropping the last row and column,  $\hat{K}$  becomes negative definite. Hence  $\text{sign}(\det(\hat{K})) = (-1)^{L-1}$ .

Consider two agents,  $i$  and  $j$ , for which for some good  $l < L$   $v_l^i > v_l^j$ . (We can always renumber the goods so that  $l$  is not the last good.) For all  $h$ , let

$$\delta^h = (x^h - e^h) - c_h (K_l)^T$$

where  $K_l$  is the  $l$ th row of  $K$ . Let  $c_i = -c_j = c > \frac{1}{v_l^i - v_l^j}$ , and let  $c_h = 0$  for all  $h \neq i, j$ . Then by Walras Law and Slutsky,  $p \cdot \delta^h = 0$  for all  $h$ . Moreover

$$D\hat{x} = \hat{K} - c(\hat{v}^i - \hat{v}^j)\hat{K}_l$$

This matrix must have determinant of the opposite sign of the determinant of  $\hat{K}$ . Its  $l$ th row is a negative scalar multiple  $1 - c(v_l^i - v_l^j) < 0$  of the row  $\hat{K}_l$ , which by itself flips the sign of  $\det(\hat{K})$ . Its other rows  $k \neq l$  are equal to their original rows  $\hat{K}_k$  plus a scalar multiple of the  $l$ th row, which induces no further change in the determinant.

Now for any scalar  $\alpha$ , define  $\alpha^+ = \max(0, \alpha)$  and  $\alpha^- = \max(0, -\alpha)$ . For any two agents  $h, k$ , and good  $g = 1, \dots, L$ , define

$$d_g^{hk} = (s_g^k)(\delta_g^h)^-$$

where  $s_g^k$  is the share among all agents of good  $g$  owed to agent  $k$

$$s_g^k = \frac{(\delta_g^k)^+}{\sum_h (\delta_g^h)^+}$$

From the fact that  $\sum_h \delta^h = 0$ , we deduce that indeed  $\delta^h = \sum_{i \neq h} d^{ih} - \sum_{i \neq h} d^{hi}$  and  $p \cdot \delta^h = 0$  for all  $h$ .

The Walrasian economy with debt  $(u^h, e^h, d^{hi})_{h,i \in H}$  has the old equilibrium  $(p, (x^h)_{h \in H}, (r^h = 1)_{h \in H})$ . In a neighborhood  $N$  of the equilibrium prices  $p$ , there is no default, so the truncated excess demand  $\hat{z}(p) = \hat{x}(p) - \sum_h e^h$  is smooth and the determinant of its jacobian  $D\hat{z} = D\hat{x}$  has sign equal to  $-(-1)^{L-1}$ , as we just saw. We now apply the Poincare-Hopf Index Theorem to deduce that there must be at least one more equilibrium.

By Lemma 10, for all small enough  $\varepsilon > 0$ , the function  $\hat{z} : P_\varepsilon \rightarrow \mathbb{R}^{L-1}$  does not point directly out on the boundary of  $P_\varepsilon = \{p \in \mathbb{R}^{L-1} : p_l \geq \varepsilon, l = 1, \dots, L-1, \sum_{l=1}^{L-1} p_l^2 \leq 1/\varepsilon\}$ . For arbitrarily small  $\eta > 0$ , we can find a smooth function  $\hat{z}_\eta : P_\varepsilon \rightarrow \mathbb{R}^{L-1}$  that agrees with  $\hat{z}$  on  $N$ , and is within  $\eta$  of  $\hat{z}$  on all  $P_\varepsilon$  and is transverse to 0. Following the argument of Dierker (1972) and Varian (1975), who appealed to the Poincare-Hopf Index Theorem, the sum of the signs of the determinants of the Jacobians of all the finite number of zeroes of  $\hat{z}_\eta$  must be  $(-1)^{L-1}$ . Since  $\hat{z}_\eta$  has one zero, at  $p$ , which has a determinant with the opposite sign, there must be at least two more equilibria with determinants of sign  $(-1)^{L-1}$ . Letting  $\eta \rightarrow 0$  and taking convergent subsequences gives at least one equilibrium of  $\hat{z}$  that is distinct from  $p$ .

Consider now debts  $d'$  close to  $d$ . Since  $D\hat{z}$  has a nonzero determinant of sign  $-(-1)^{L-1}$ , by the implicit function theorem, for  $d'$  close enough to  $d$ , the economy  $(u^h, e^h, d'^{hi})_{h,i \in H}$  also has a no-default equilibrium with determinant of the same sign. Applying the same argument from the last paragraph shows that  $(u^h, e^h, d'^{hi})_{h,i \in H}$  also has multiple equilibria. ♠

This proof formalizes the intuition given in the introduction. Recall that in the proof above, the agent  $i$  with higher marginal propensity to consume good  $l$  is given extra endowment  $\delta^i = -cK_l$ . This entails a positive amount of good  $l$ , because own substitution effects are negative,  $K_{ll} < 0$ , hence  $\delta_l^i > 0$ . Since  $p \cdot \delta^i = 0$ , there is another good  $l'$  with  $\delta_{l'}^i < 0$ . In effect agent  $i$  incurred a big debt in  $l'$  that he used to buy  $l$ . In the equilibrium at prices  $p$  he is selling  $l$ . The theorem shows that if the price  $p_l$  rises a little, the excess demand for  $l$  rises, thus pushing the price  $p_l$  still higher.

## 10 Default Multiplicity

The uniqueness of consistent delivery rates proved in Section 5 originates from the fact that seizing collateral is frictionless. It is possible however to imagine an economy where circular debt arrangements lead to delivery rate multiplicity, arising from losses associated with defaults. This section provides an example.

Consider an economy with three agents where some defaults are costly. For example say that when agent 1 defaults and is able to send only  $r_1 < 1$  of the goods that she owed, the default itself creates a proportional loss of  $1 - r_1 < 1$  of the goods sent, so that only  $r_1^2$  arrives at her creditors. Incomes are

thus

$$\begin{aligned}
I^1(p, r) &= p \cdot e^1 + \sum_{i \neq 1} r_i p \cdot d^{i1} - r_1 \sum_{i \neq 1} p \cdot d^{1i} \\
I^2(p, r) &= p \cdot e^2 + r_1^2 p \cdot d^{12} + r_3 p \cdot d^{32} - r_2 \sum_{i \neq 2} p \cdot d^{2i} \\
I^3(p, r) &= p \cdot e^3 + r_1^2 p \cdot d^{13} + r_2 p \cdot d^{23} - r_3 \sum_{i \neq 3} p \cdot d^{3i}
\end{aligned}$$

For simplicity let all agents have identical linear preferences so that prices are fixed at some  $p$ . This implies that we can limit the description to endowment and debt values. Let the endowments values be

$$p \cdot e^1 = 2, \quad p \cdot e^2 = 0, \quad p \cdot e^3 = 6$$

and debt values be

$$\begin{array}{ccc}
- & p \cdot d^{12} = 16, & p \cdot d^{13} = 0, \\
p \cdot d^{21} = 15, & - & p \cdot d^{23} = 3 \\
p \cdot d^{31} = 0, & p \cdot d^{32} = 3, & - .
\end{array}$$

The delivery vector of  $r = (1, 1, 1)$  leads to positive income for all agents,

$$I^1 = 1, \quad I^2 = 1, \quad I^3 = 6,$$

so it is self consistent. But the delivery vector of  $r = (\frac{3}{4}, \frac{2}{3}, 1)$  is also self consistent. Because  $r_1 = \frac{3}{4}$  agent one sends only 12 to agent two. But only  $\frac{3}{4}$  of that survives the journey, so agent two gets only 9 worth of goods. Because  $r_3 = 1$  agent two has 3 worth of goods delivered from agent three, so with 12 units in total coming in, no endowment and debt of 18, it must default and only deliver  $r_2 = \frac{2}{3}$  of the goods that it owes. Agent one now has an endowment valued at 2 and delivery from agent two valued at 10. With a debt of 16 it must default and deliver only  $r_1 = \frac{3}{4}$  of the goods that it owed. Agent three has an endowment of 6, from which it must deliver 3 and only receive 2. Incomes are now

$$I^1 = 0, \quad I^2 = 0, \quad I^3 = 5,$$

with the cost of the 3 units lost in delivery from agent one to agent two spread across all agents.

In a pure endowment economy multiple equilibria never Pareto dominate each other. This is also true with default. When some agents default there must be some other agents that ultimately consume

the defaulting agents' endowment. So prices only transfer wealth across agents. In Diamond and Dybvig (1983) the two equilibria are Pareto ranked because by assumption the act of transferring ownership disrupts production. The argument of this paper has been that multiplicity does not require any of the special assumptions of Diamond and Dybvig (1983), just debt. What the example of this section shows is that it is possible to also get Pareto ranked multiplicity, but we do need to assume some way for the economy to lose goods. As to what assumption better describes the conditions that trigger the loss of goods, it remains an empirical question. Our conjecture is that the inefficiency of bankruptcy proceedings is much larger than the production disruption caused by a normal change of ownership.

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