Blood Allocation with Replacement Donors: A Theory of Multi-unit Exchange with Compatibility-based Preferences*

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Abstract

In 56 developing and developed countries, blood component donations by volunteer non-remunerated donors can only meet less than 50% of the demand. In these countries, blood banks rely heavily on replacement donor programs that provide blood to patients in return for donations made by their close relatives or friends. These programs appear to be disorganized, non-transparent, and inefficient. We introduce the design of replacement donor programs and blood allocation schemes as a new application of market design. We formulate a general blood allocation and replacement donation model. Within this framework, we introduce optimal blood allocation mechanisms that accommodate fairness, efficiency, and other allocation objectives, together with endogenous exchange rates between received and donated blood units beyond the classical one-for-one exchange. Additionally, the mechanisms provide correct incentives for the patients to bring forward as many replacement donors as possible. This framework and the mechanism class can also apply to general applications of multi-unit exchange of indivisible goods with compatibility-based preferences beyond blood allocation.

Keywords: Blood transfusion, market design, multi-unit exchange, dichotomous preferences, endogenous pricing

JEL Codes: D47, C78, I12, I19, D82, D78

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1 Introduction

Transfusions are commonly used to treat various medical conditions to replace lost blood or add inadequate blood components. Replacement red blood cells and other blood components such as platelets, plasma, and clotting factors are essential for patients going through certain procedures such as surgery, chemotherapy, and child birth and for patients with trauma and blood diseases.[1] In the US, according to Pfuntner, Wier, and Stocks (2013), blood transfusion was the most common procedure performed during hospitalizations in 2011. While blood transfusion saves lives and improves health outcomes, many patients requiring transfusion around the world do not have timely access to safe blood due to significant supply shortages.

Around the world, the collection and distribution of blood is organized through blood banks where donated blood is processed and stored. Unlike most solid human organs and tissues, blood replenishes after donation and most blood products can be stored for a period of time. Thus, a healthy donor can donate whole blood regularly once in every two months and some components, such as platelets and plasma, more frequently. Different compatibility requirements apply for each blood component (see Section 2 for medical and institutional details of blood component transfusion including various compatibility requirements).

The most adequate and reliable supply of blood is through *volunteer non-remunerated donors* (*VNRDs*), who mostly donate blood, often repeatedly, through blood drives or other campaigns.[2] These donors provide the safest supply of blood, since the prevalence of blood-borne infections is lowest among this group of donors.[3] According to the World Health Organization (WHO), 79 countries (38 high-income, 33 middle-income, and 8 low-income) collect more than 90% of their blood supply from VNRDs (WHO, 2020). The World Health Assembly resolution WHA63.12 (Sixty-third World Health Assembly, 2010) urges all member states to develop national blood systems based on VNRDs and to work toward the goal of self-sufficiency. Despite these warnings, donations by VNRDs remain insufficient to meet the demand for blood and its components in many regions of

[1]Since most patients require a specific blood component for treatment, whole blood is rarely used in modern transfusion medicine except in some low-income countries (WHO, 2020).

[2]Blood is forbidden to be exchanged using valuable remuneration in most countries. Nevertheless, it is reported that 16 countries collect blood through paid donations as of 2018 (WHO, 2020).

[3]Paid donors are considered to be inferior, as they may be in poorer health than VNRDs. Such donors may also have incentives to hide their health status, causing adverse selection problems. Paid donation is allowed in the US, as blood is not covered by the National Organ Transplant Act of 1984, which forbids the sale of solid human organs and tissues. In spite of this fact, in the US blood components used for transfusion are obtained almost exclusively from VNRDs because of safety and ethical concerns.

the world.

Although it seems relatively costless to donate blood, there are severe blood shortages in many developing countries, as well as seasonal shortages in developed countries (Gilcher and McCombs, 2005).[4] Cultural and religious factors create frictions that deter VNRDs, especially in some developing countries. Furthermore, some blood components, such as platelets, have short shelf life, are in high demand, and are more difficult to collect than the others. Thus, shortages of such components occur even in the developed world.

In 56 countries worldwide (9 high-income, 37 middle-income, and 10 low-income), more than 50% of the blood supply is met by *replacement donors* and, in some cases, through paid donors (WHO, 2020). As an effective method to boost blood component reserves, blood banks in many places—including highly populated countries such as India, China, and Brazil—employ official or unofficial *replacement donor programs*. A replacement donor program requires each patient to nominate a number of willing donors, who are typically family members or close friends, to donate in order for the patient to receive transfusion.[5]

Notwithstanding the important role they play in addressing blood shortages, existing replacement donor programs suffer from two major shortcomings.

The first shortcoming is the loss of welfare due to the lack of optimized inventory management based on donor screening and the needs of the blood bank. Although inventory management is often considered among the most important goals for a blood bank, as far as we know, no explicit optimization is pursued in current replacement donor programs to achieve certain policy objectives. In the face of chronic supply shortages, one such natural objective can be to maximize the allocated blood volume using the correct set of replacement donors.

The second shortcoming is that replacement donor programs generally operate on

[4]There are often shortages of type O red blood cells in the US in the early winter and midsummer months. Outside of seasonal factors, blood shortages can often frequently occur during catastrophic events such as earthquakes or pandemics. For example, during the recent COVID-19 pandemic, blood components have had shortages in the US (American Red Cross, 2020a).

[5]Within the medical community, there is an ongoing debate about the stance of the WHO regarding VNRDs being the safest blood supply. There has been considerable evidence suggesting that the blood collected through replacement donors is as safe as VNRDs. It is also argued that the motivations of the two types of donations are similarly altruistic, and the distinction between them from an ethical perspective is not clear cut. Allain and Sibinga (2016) provide an excellent survey of these views, empirical evidence, and references. In addition, there are significant economic and cultural reasons for the predominance of decentralized and often hospital-based replacement systems in many developing countries. Such a system is much less costly (Bates, Manyasi, and Lara, 2007), favors intra-group solidarity, and is culturally more consistent with the presence of strong family or community bonds (Haddad, Bou Assi, and Garraud, 2018; Kyeyune-Byabazaire and Hume, 2019).

fixed exchange rates between units (of blood) received by the patient and units supplied by the patient's donors, which creates issues of efficiency, fairness, and ethics. Certain patients may not be able to recruit the required number of donors that they are obliged to provide, making it difficult to receive blood. The rules of replacement donor programs are sometimes bent arbitrarily in favor of such patients, or such patients pay third parties to assume the role of their replacement donors creating black markets. Additionally, around the world, replacement donor programs appear to be highly non-transparent in their blood allocation operations. It is difficult to find existing guidelines that govern these processes (see Section 2 for institutional details of how real-life replacement donor programs function). Even in the absence of these problems, a fixed exchange rate regime limits the scope of admissible exchanges and allocations.

In this paper, we introduce blood allocation with VNRDs and replacement donors as a novel market design problem and propose a blood allocation model together with solutions to address these shortcomings. In the model, each patient has a *maximum need* of blood that is usually determined by her medical condition. In addition, the blood bank provides her a *minimum guarantee* that can be set at the minimum need of the patient or at zero during severe shortages. Each patient brings forward a (possibly empty) set of replacement donors. We assume that each donor, who is represented by her blood type, can donate one unit of blood without loss of generality. The blood bank also has an inventory of blood of each type, which can be interpreted as coming from VNRDs.

The blood bank chooses a blood-type compatible allocation depending on the needs of patients, the availability of replacement donors, and its inventory. The allocation specifies, for each patient, the amount of blood of each compatible blood type she receives and which of her donors donate. Each patient's welfare is determined by the *schedule* induced by this allocation, which is represented by the total amount she receives and the total amount she supplies through her replacement donors. Naturally, each patient is assumed to have lexicographic preferences: she prefers receiving more blood to less; given a certain amount of blood received, she prefers to supply less (see Section 2.2).

To accommodate various blood transfusion and replacement donor protocols, we introduce the notion of a *feasible schedule correspondence*. This idiosyncratic correspondence of each patient specifies all possible schedules the patient can be assigned under an allocation for each set of donors provided by the patient. In particular, a patient does not necessarily supply one unit for each unit of blood received (the classical one-for-one exchange), as a flexible menu of possible schedules can be designated by such a correspondence. We view the design of feasible schedule correspondences as an important policy variable and novelty in the paper.

Then we propose and study a general class of *optimal mechanisms*. Each optimal mechanism is represented by the maximization of an *additively responsive* aggregate preference relation over schedule profiles of the patients, subject to feasibility constraints designated by the feasible schedule correspondence of each patient, as well as market clearing and blood-type compatibility conditions (see Section 4). This class includes practical mechanisms that fulfill the blood bank's various allocation and inventory management objectives, such as *sequential targeting mechanisms* (that maximize the amount of blood received or minimize the amount of blood supplied by each target patient group in a sequential manner) and *weighted maximal mechanisms* (that maximize the difference between a weighted sum of the amounts received by the patients and a weighted sum of the amounts supplied). Optimal mechanisms also nest all previously studied mechanisms for the multi-unit exchange of indivisible goods with compatibility-based monotonic preferences as special cases (see Section 6).

The optimal mechanisms together with the feasible schedule correspondences overcome the two shortcomings of current replacement donor programs outlined above.

First, they address the lack of optimization based on donor screening. In particular, the optimal mechanisms are efficient for patients under basic alignment conditions of the aggregate preference relation over schedule profiles with patients' preferences (Remark 1). They are also *donor monotonic*, i.e., providing a larger set of donors does not reduce the amount of blood the patient receives, under three natural restrictions on the feasible schedule correspondences (Theorem 2): every feasible schedule set satisfies a discrete convexity notion, *L(attice)-convexity*; each unit of blood has a positive "price," i.e., receiving more blood requires more donations in terms of feasibility; and the feasible schedule set becomes more favorable for the patient as her donor set expands. Among these conditions, L-convexity plays an important role, which also guarantees that the outcome of a weighted maximal mechanism can be found in polynomial time (see Appendix C.2 in Supplemental Material). Achieving donor monotonicity is particularly important in this context as it helps align patients' individual incentives with the blood bank's objective of increasing blood transfusion. We show that optimal mechanisms satisfy a stronger incentive compatibility notion when the last restriction on the feasible schedule correspondences is strengthened (Theorem 3).[6]

Second, the innovation of feasible schedule correspondences allows for various ex-

[6] We also provide comparative static analysis for changes in feasible schedule correspondences (Theorem 4).

change rates between units received and supplied, while optimal mechanisms determine endogenously these exchange rates. This property helps rectify the shortcoming caused by a fixed exchange rate in current programs, as these feasible schedule correspondences can be tailored fairly for patients who can intrinsically recruit fewer donors, or for different medical conditions, which help prevent black markets. As a result, our approach provides a framework to assess and improve the effectiveness of the existing replacement donor programs, and makes it possible to offer rigor and transparency to their organization. Toward this goal, we provide concrete policy designs and implementation proposals (see Section 5). We also conduct simulations to show the possible gains from our design. Under a set of realistic parameters, a sequential targeting mechanism under flexible exchange rates leads to 19%-28% more transfusions than the same mechanism under the one-for-one exchange rate, which in turn leads to 164% to 3% more transfusions than an emulation of current replacement donor practices.

Unlike the living-donor organ exchanges that have attracted much attention in the last two decades in both the market design literature and practice, blood allocation involves multi-unit demand and supply.[7] Moreover, many other factors make this market design problem theoretically and practically different from the analysis and functioning of solid organ exchanges. These include differences in the compatibility requirements for different blood components, the possibility of endogenous and non-unit exchange rates between blood received and supplied, the non-simultaneity between donation and transfusion, and the possibility to store blood components.

Our model and theoretical results are independent of the particular background of blood allocation and can readily be applied to other contexts with a subset of similar features within the framework of multi-unit exchange of indivisible goods with compatibility-based monotonic preferences in units consumed. Although compatibility is verifiable in blood allocation, there can be other contexts where this is private information for each agent. Some such applications studied in the literature include shift exchanges among the workers in a company (Manjunath and Westkamp, 2021) and time banks and favor exchanges (Andersson et al., 2021). We also extend our analysis to this general domain in Appendix B in Supplemental Material and consider the incentives to truthfully reveal compatibility relation as well as endowment. We show that optimal mechanisms are

[7]See Sönmez and Ünver (2017) for a recent survey of this literature and the practical developments. Notable exceptions to unit-demand organ exchanges are living dual-donor lobar lung transplantation, dual-graft living-donor liver transplantation, and simultaneous liver-kidney transplantation (Ergin, Sönmez, and Ünver, 2017). However, no organized exchange program exists for these practices as of writing of this paper.

weakly strategy-proof under our baseline assumptions: no agent receives more compatible units by misreporting her compatibility relation and/or under-reporting her endowment set. Under more stringent conditions, we show that they are fully strategy-proof. Thus, our mechanisms and incentive results substantially generalize and subsume previous ones under compatibility-based preferences. Moreover, as far as we are aware, all previous exchange mechanisms in the literature use the exogenous one-for-one exchange rate. As an important theoretical contribution, we overcome this limitation and introduce endogenous pricing of units while maintaining the good incentive properties of the mechanisms (see Section 6 for more on this and other related literature).

2 Background

2.1 Main Blood Components and Compatibility

There are different transfusion protocols for different blood components, and the medical practices also vary across different regions of the world. We mainly focus on the three most-transfused blood components—red blood cells, platelets, and plasma—as well as whole blood, and provide a brief account starting with a general rule of thumb for compatibility requirements.

Blood-type compatibility plays an important role for the feasibility of transfusion. There are more than 300 human blood groups. Two of them are the most important in clinical practices. The first one, the ABO blood group system, is the most commonly known. A person's ABO blood type is determined by the presence of A or B antigens in her blood cells: her type may be O (if she has neither antigen), A (has only the A antigen), B (has only the B antigen), or AB (has both antigens). Each person has pre-formed antibodies in her plasma against every non-existent antigen. Antibodies against an antigen attack blood cells that carry this antigen, which can cause potentially fatal hemolysis.

Therefore, any transfusion including a significant amount of donor cells, by rule of thumb, should respect *ABO-cellular compatibility*: O blood-type cells can be donated to all, A blood-type cells can be donated to A and AB blood-type patients, B blood-type cells can be donated to B and AB blood-type patients, and AB blood-type cells can only be donated to AB blood-type patients.

On the other hand, any transfusion including a significant amount of donor plasma, which carries the donor's pre-formed antibodies, by rule of thumb, should respect *ABO-plasma compatibility*: AB blood-type plasma can be donated to all as it does not contain any antibodies, A blood-type plasma can be donated to A and O blood-type patients, B

blood-type plasma can be donated to B and O blood-type patients, and O blood-type plasma can only be donated to O blood-type patients as it contains antibodies against both antigens.

The second crucial blood group system is Rh. The most clinically important Rh antigen is D. Its existence and non-existence correspond to Rh D+ type and Rh D− type respectively. Antibodies to the Rh D antigen can only develop on an Rh D− person after being exposed to Rh D+ red blood cells. Hence, the compatibility requirement is to avoid the transfusion of Rh D+ red blood cells to an Rh D− patient, due to the risk of hemolytic reactions.

Most blood components are packed with others in solutions. Thus, depending on the amount of these components, different practices are followed for the compatibility of the pack with the patient.

Next, we turn our focus to specific blood components.

Red Blood Cells

Red blood cells carry oxygen from the lungs to all parts of the body and are the most commonly transfused blood components. Red blood cell transfusion—the de-facto modern day replacement for the older whole blood transfusion therapy—is mostly used for patients with anemia due to cancer, blood diseases, and other causes, followed by surgical patients. Whole blood is still transfused in some low-income countries. For other countries, this is only occasionally performed in emergencies for patients with massive blood loss due to trauma, surgeries, etc. A person donates one unit (about a pint) of whole blood each time and she has to wait at least eight weeks between donations. Each unit of red blood cells is prepared from one unit of donated whole blood by removing plasma and adding preservative solutions, and can be stored for about 42 days.

ABO-identical and Rh D-compatible transfusion is generally practiced for whole blood transfusion.[8] For red blood cells, ABO-cellular compatible and Rh D-compatible transfusion is all that is needed in theory. However, as red blood cell packs usually carry some amount of donor plasma, ABO-identical (and Rh D-compatible) transfusion is often required.

Eight blood types are relevant for red blood cell or whole blood transfusion. However, in some populations, such as those in Asia, Rh D− is so rare that there are effectively only four blood types.[9]

[8] An exception is that type O Rh D− blood is often transfused in emergencies to patients with other or unknown blood types. For this reason it is also dubbed as the *global-donor* blood type.

[9] For example, in China, the Rh D antigen exists in more than 99% of the population.

Platelets

Platelets are tiny cells in the blood that form clots and stop bleeding. Platelet transfusions are mostly given to prevent or treat bleeding in patients with thrombocytopenia or abnormal platelet function, such as those undergoing chemotherapy or receiving a bone marrow transplant. McCullough (2010) states that the use of platelets has increased more than other blood components in the last 15 years. According to Red Cross of America, every 15 seconds someone needs platelets (American Red Cross, 2020b). However, due to their storage requirement at room temperature, platelets have a much shorter shelf life than most other blood components: in most countries they can only be stored between four and seven days (Cid, Harm, and Yazer, 2013). As a result, platelets are in frequent shortages even in developed countries.

One unit of platelets can be prepared from 4-6 units of pooled whole blood, or obtained from a single donation through the technique of *apheresis*, which only takes platelets out of the donor's blood, leaving the other components in the blood stream. The whole process takes approximately three hours and a person can donate platelets in this way once a week, up to 24 times a year.[10] In addition to the efficiency in the production process, apheresis platelets are also safer to the patients due to the minimal donor exposure. Hence, it has become an increasingly common practice to give apheresis platelets, instead of whole-blood-derived platelets. In 2017, only 4.2% of the total transfused platelet units in the US were derived from whole blood (Jones et al., 2020).[11]

For platelets, the compatibility practices vary significantly among different institutions and countries. As platelets (weakly) express the ABO antigens and they are suspended in plasma in the platelet packs, ABO-identical transfusion is always preferred, although ABO incompatibility in platelet transfusion is generally not as risky as in whole blood or red blood cell transfusion. Given the frequent shortages, ABO-identical transfusion is often not possible. Both ABO-cellular compatible transfusion and ABO-plasma compatible transfusion are commonly practiced, and there has been no consensus as to which is the better strategy (Dunbar et al., 2015; Lozano et al., 2010; Norfolk, 2013). Finally, as the Rh D antigen is not present on platelets, Rh D compatibility is usually not required (for example, see Cid, Harm, and Yazer, 2013).

Plasma

Plasma is the non-cellular, protein- and antibody-rich liquid component of blood. The

[10] A donor usually donates one unit of platelets through apheresis, but double or triple-unit donation in a session may also be possible, depending on the health of the donor.

[11] The apheresis method has also become popular in developing countries (Eichbaum et al., 2015).

plasma used in everyday transfusion is usually *fresh frozen plasma*. Plasma transfusion is often utilized by patients with liver failure, heart surgery, severe infections, and serious burns. One unit of fresh frozen plasma can be prepared from one unit of whole blood after removing the blood cells. Alternatively, a person can donate up to three units through apheresis, which keeps other blood components in her blood stream and only extracts plasma. Fresh frozen plasma has the longest shelf life among the three main blood components: it can be stored for about a year. Its transfusion follows ABO-plasma compatibility, without regard to Rh D compatibility (as Rh D antibodies only form after exposure to the Rh D antigen and are not pre-formed).

Convalescent plasma, the antibody-rich plasma of a patient recovering from an infectious disease with no other known cure, such as Ebola and most recently COVID-19, is commonly used to treat patients or to produce drugs against the disease. It can also be considered as a type of fresh frozen plasma.

In addition to plasma used for transfusion, plasma derivatives (such as albumin, coagulation factors, and immunoglobulins) manufactured from "source plasma" in fractionation centers are used in the treatment of various conditions. Unlike the blood used for transfusion, source plasma is commonly collected from paid donors in many countries.[12]

2.2 Blood Demand of a Patient

The amount of a blood component needed to treat each medical condition is idiosyncratic. For example, Collins et al. (2015) report that, at a tertiary referral center in the US, the average amount of red blood cell units used per surgery is close to 3.5 units and this amount has a high variance due to various patient conditions.

Besides the idiosyncratic demand, there is usually a range of units where each amount in the range can be transfused to a given patient. However, receiving more units can be better under various outcome or preference metrics. We give three general examples of patient demand that have this common thread.

First, it is medically acceptable and feasible to transfuse a range of units to a patient with a particular condition such that more units lead to better outcomes. For example, platelets are often transfused prophylactically to prevent bleeding when a patient's platelet count is below a certain threshold. In such cases, both the strategy of higher doses in lower frequency and the strategy of lower doses in higher frequency are practiced (Stroncek and Rebulla, 2007). Norol et al. (1998) show that the high and very high dose

[12]The US has a large source plasma industry that relies on paid donors, and it is responsible for 55% of the world's supply of plasma derivatives (Farrugia, Penrod, and Bult, 2010).

treatments lead to significantly better platelet increment in the patients, compared to the medium dose treatment.

Second, the exact need of a patient can be ex-ante uncertain. For example, there may be some minimum and maximum possible units to be transfused during a surgery. For cautionary reasons, a surgeon often orders significantly more blood than the patient ends up using. Collins et al. (2015) report that 72% of the red blood cells ordered for surgeries go unused. The ratio of ordered to transfused red blood cells can be as high as 11 to 1 in elective liver resection surgical procedures (Cockbain et al., 2010). These ratios indicate that surgeons are quite risk averse. Indeed, Collins et al. (2015) note that surgical blood loss can be unpredictable, so some leeway for ordering red blood cells that ultimately go unused is necessary for safe patient care. Ex-ante a surgeon has monotonic preferences over the amount of blood ordered as long as the amount exceeds some minimum threshold.

Third, blood components such as platelets and red blood cells are often transfused routinely to patients with chronic conditions and are administered in small doses over time. For example, Marwaha and Sharma (2009) state that patients undergoing chemotherapy require platelet transfusion once in at least every three days, and, when the bone marrow is adversely affected, every day. In such cases, more units are preferred to less in a time interval, although several transfusions can be conducted in this interval.

2.3 Blood Bank Policies for Replacement Donation

Replacement donor programs are observed in all continents and are especially common in Africa, Latin America, and Central Asia (Allain and Sibinga, 2016). Populous countries such as Pakistan, Brazil, and Mexico collect their blood components almost entirely through replacement donor programs. On the other hand, countries such as India and China rely on these programs to meet the demand not met by VNRDs. A patient's replacement donors can donate before or after the patient receives blood depending on the regional practice. Since direct donation from a donor to the patient (even if they are compatible) is not practiced in modern medicine due to health concerns (i.e., the donor blood needs to be tested and processed first), the blood bank is used as an intermediary.

Blood banks work with hospitals and blood centers. Hospitals relay the needs of patients to the blood banks, while the blood banks and blood centers collect donations from VNRDs and replacement donors. Hospitals are often required to maintain a small inventory of their own (for example, see Delhi State Health Mission, 2016).

Although replacement donor programs are very common and officially acknowledged in many countries, maybe surprisingly, it is difficult to find their exact institutional details. The most common practice in current replacement donor programs worldwide is that the blood bank announces, either officially or unofficially, a preset exchange rate between the units of blood received and supplied, often irrespective of the blood type sought or donated. Blood banks provide blood to patients exclusively based on these rates. Among these, the one-for-one exchange rate, i.e., one unit replacement per unit received, is most common around the world. We next give some examples of policies practiced by replacement donor programs, with a particular focus on non-unit exchange rates.

Although China banned the replacement donor programs in 2018, they are still used in several cities during shortages, especially for platelets (She, 2020). Different policies have been in place in different localities during the official phase and the current phase. In most cities, including Beijing, the exchange rate is one-for-one. As reported by She (2020), in Xi'an, during periods of shortages, a patient has the priority of receiving three units of blood for every unit she has donated before, and she has the priority of receiving one unit for every unit her replacement donors donate now.[13] In Guangzhou, there is not necessarily a fixed relation between the amount received and supplied (Chen, 2012). Moreover, according to Chen (2012), in some regions there are restrictions on the blood types of replacement donations. As an extreme case, the blood type of a replacement donor must be identical to that of the patient in Jiangsu. While such a restriction is relatively rare for whole blood donations, it is not uncommon for replacement platelet donations throughout the country.

India has the largest official replacement donor programs in the world after Pakistan. In Delhi, regardless of the amount of blood she needs, the patient is required to bring forward one replacement donor, unless the intervention needed is an emergency surgery (Delhi State Health Mission, 2016).

In Cameroon and Congo, the exchange rate has been two replacement units per unit received, as almost 25% of the donations are not suitable for transfusion due to infections (Tagny, 2012). The same exchange rate is also used in Puerto Vallarta, Mexico, for cost reasons (Thompson, 2020).

In Tucuman, Argentina, a patient's replacement donors donate after the transfusion. The exchange rate is fixed at one-for-one; however, it is not as strictly enforced.[14]

[13] In practice, these so-called priorities essentially ensure that the patient can receive the blood.

[14] Based on personal communication with the director of the Tucuman Blood Bank, Dr. Felicitas Agote, on July 7, 2020.

2.4 Institutional Constraints

The feasibility of blood transfusion primarily depends on the blood type compatibility. Therefore, replacement donor programs operate on the premise of exchange of willing donors for compatible blood received by the paired patient. This is similar in principle to organ exchanges with the first-order difference that there is not yet an optimized central clearinghouse for replacement donors. There are a number of other important institutional differences. To begin with, the logistical constraints of blood donation are negligible compared to those in organ transplantations. The blood donation process takes only a few hours and its effects wear off relatively quickly. On the other hand, organ transplantations carry risks and require careful planning weeks before and after the operations. Once extracted, blood components can be stored for a certain period of time, which can facilitate the designer's choice of optimal timing of assignments. Moreover, many blood banks and hospitals often operate in coordination, making it possible to obtain the necessary blood units from neighboring facilities. These lead to the observation that in blood allocation with replacement donors, the possibility of a donor reneging is not as much of a concern as in organ exchanges.[15]

The logistical ease and flexibility in blood allocation have led to different and innovative incentivization schemes to promote blood donation. The assignment of voucher credits has been a popular approach in practice. For example, blood assurance programs in the US guarantee each VNRD or her tax-code dependents exactly the same amount of blood donated in the event of a future need.[16] Similar programs have also been traditionally implemented in China. Recently, Kominers et al. (2020) proposed a similar incentive scheme for COVID-19 convalescent plasma donation.[17] Replacement donor programs differ from these proposals, as we are considering the improvement of existing programs that usually do not have many voucher or memory features, nor the pay-it-backward or pay-it-forward features discussed in the literature. Thus, blood allocation is more in line with the analogy of organized organ exchanges without simultaneity or other severe constraints.

[15]For the readers who are familiar with the organ exchange literature in market design, the absence of logistical constraints together with the ability to store blood components make it possible to incorporate cycles and chains of arbitrary length into an allocation in our problem, unlike organ exchanges.

[16]An example is the program of Cape Fear Valley Blood Bank in North Carolina (Cape Fear Valley, 2020).

[17]A voucher-based scheme is also used for kidney exchange in the US (Veale et al., 2017), and it has been proposed for compatible pairs to participate in kidney exchange (Sönmez, Ünver, and Yenmez, 2020).

3 The Model

We consider the market for a single blood component, which we simply refer to as blood.[18] Let I be a set of **patients** and $\mathcal{B}$ be the set of **blood types**.[19] Each $X \in \mathcal{B}$ denotes a specific blood type used in compatibility requirements. Each patient $i \in I$ has type $\beta_i \in \mathcal{B}$ blood and needs a **maximum** of $\overline{n}_i \in \mathbb{Z}_{++}$ units of blood. For each $X \in \mathcal{B}$, $\mathcal{C}(X) \subseteq \mathcal{B}$, $\mathcal{C}(X) \neq \emptyset$, is the set of blood types compatible with a type X patient. Each patient i also has a (possibly empty) set of willing **replacement donors** D_i such that each donor $d \in D_i$ can provide one unit of type $\beta_d \in \mathcal{B}$ blood. Let $\mathcal{D}_i$ be the collection of all possible donor sets that a patient $i \in I$ can bring forward. Assume that if $D_i \in \mathcal{D}_i$ and $D_i' \subseteq D_i$, then $D_i' \in \mathcal{D}_i$. Let $\beta_I = (\beta_i)_{i\in I}$, $\beta_D = (\beta_d)_{d \in \cup_{i\in I} D_i}$, $\overline{n} = (\overline{n}_i)_{i\in I}$, $D = (D_i)_{i\in I}$ and $\mathcal{D} = \Pi_{i\in I}\mathcal{D}_i$.

The **blood bank**, denoted as b, has v_X units of type X blood in its **inventory** for each $X \in \mathcal{B}$. Let $v = (v_X)_{X\in\mathcal{B}}$. The blood bank guarantees a **minimum** of $\underline{n}_i \in \{0, 1, \ldots, \overline{n}_i\}$ units of blood for each patient $i \in I$ if the patient participates in the replacement donor program, i.e., if she provides a qualified set of donors (we will explain the quantifications of the donor set to qualify for the minimum guarantee below when we introduce feasible schedules). Let $\underline{n} = (\underline{n}_i)_{i\in I}$. Assume that for any non-empty subset of blood types $\mathcal{B}' \subseteq \mathcal{B}$,

$$\sum_{i\in I:\beta_i\in\mathcal{B}'} \underline{n}_i \leq \sum_{X\in\cup_{Y\in\mathcal{B}'}\mathcal{C}(Y)} v_X.$$

Therefore, the blood bank carries enough blood to meet the minimum guarantees regardless of the replacement donors that will be brought by the patients.[20] Generally, the minimum guarantee profile is a policy variable determined by the blood bank based on its inventory. A patient's minimum guarantee may be related to her medical condition and correspond to the minimum threshold needed to treat her condition. It can also be based on the good samaritanship of the patient in the past.[21] The minimum guarantees may be set to zero during severe shortages.

[18]In Appendix E.2 in Supplemental Material, we discuss how to integrate different blood component markets.

[19]For example, in whole blood transfusion, $\mathcal{B} = \{O+, O-, A+, A-, B+, B-, AB+, AB-\}$. The plus sign "+" and the minus sign "−" represent Rh D+ and Rh D−, respectively.

[20]Specifically, it follows from Hall's Theorem (Hall, 1935) that this assumption is necessary and sufficient for the blood bank to be able to provide $\underline{n}_i$ units of compatible blood to each patient i using its inventory.

[21]For instance, as mentioned before, some countries use blood assurance (such as the US) or voucher programs (such as China) so that a patient who has donated blood in the past can receive credits for transfusions. Thus, she may be covered up to a certain amount of blood, even if she does not currently bring forward any donors.

Since each patient demands and (possibly) supplies blood through her replacement donors, we impose restrictions on the relationship between the amount of blood received and the amount of blood supplied. A **schedule** is a pair of non-negative integers (r, s), where r denotes the amount of compatible blood received and s denotes the amount of blood supplied. For every patient $i \in I$, her **feasible schedule correspondence** $\mathcal{S}_i$ assigns a non-empty set of schedules $\mathcal{S}_i(D_i)$ to each donor set $D_i \in \mathcal{D}_i$ such that

- $\mathcal{S}_i(D_i) \subseteq \{0, \underline{n}_i, \ldots, \overline{n}_i\} \times \{0, \ldots, |D_i|\}$, and
- $\mathcal{S}_i(D_i) = \{(0,0)\}$ or $\min\{r : (r,s) \in \mathcal{S}_i(D_i)\} = \underline{n}_i$.

The definition of a feasible schedule correspondence captures possible complementarities between units received and supplied. If $\mathcal{S}_i(D_i) = \{(0,0)\}$, the patient's donor set does not meet the minimum requirement by the blood bank for participating in the program or receiving her minimum guarantee. On the other hand, if $\min\{r : (r,s) \in \mathcal{S}_i(D_i)\} = \underline{n}_i$, then the donor set D_i satisfies the requirement, and thus, the bank is obliged to give the patient at least $\underline{n}_i$ units of blood. Let $\mathcal{S} = (\mathcal{S}_i)_{i \in I}$.

A feasible schedule correspondence is an important policy lever of the blood bank. Before proceeding further with the model, it is useful to emphasize the flexibility and the generality of our setup through the following example. In this example, we model some policies used around the world as possible feasible schedule correspondences.

Example 1. Real-Life Policy Examples:

- **The standard one-for-one policy:** Many replacement donor programs use one unit supplied per unit received exchange rate. This leads to the following feasible schedule correspondence: for every $D_i \in \mathcal{D}_i$,
$$\mathcal{S}_i(D_i) = \begin{cases} \{(0,0)\} & \text{if } |D_i| < \underline{n}_i \\ \{(r,s) \in \mathbb{Z}_+^2 : s = r \text{ and } \underline{n}_i \le r \le \min\{\overline{n}_i, |D_i|\}\} & \text{otherwise} \end{cases}.$$
- **Delhi policy:** According to guidelines (Delhi State Health Mission, 2016), each patient has to register one donor regardless of the amount of blood she needs. This can be modeled as the following feasible schedule correspondence: for every $D_i \in \mathcal{D}_i$,
$$\mathcal{S}_i(D_i) = \begin{cases} \{(0,0)\} & \text{if } D_i = \emptyset \\ \{(r,s) \in \mathbb{Z}_+^2 : 0 \le s \le 1, \text{ and } \underline{n}_i \le r \le \overline{n}_i\} \setminus \{(0,1)\} & \text{otherwise} \end{cases}.$$
- **Cameroon, Congo, and Mexico policies:** For each unit of blood received, two units of blood have to be supplied (Tagny, 2012; Thompson, 2020). Therefore, for

every $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \begin{cases} \{(0,0)\} & \text{if } |D_i| < 2\underline{n}_i \\ \{(r,s) \in \mathbb{Z}^2_+ : s = 2r \text{ and } \underline{n}_i \le r \le \min\{\overline{n}_i, \lfloor |D_i|/2 \rfloor\}\} & \text{otherwise} \end{cases}.$$

- **Xi'an, China policy:** A patient is guaranteed three units for each unit she has donated before, and the exchange rate is one-for-one beyond this guarantee (She, 2020). Let $x_i \in \mathbb{Z}_+$ be the amount of previous donations from the patient.[22] Then, her feasible schedule correspondence is as follows.
  If $\overline{n}_i \le 3x_i$, then for every $D_i \in \mathcal{D}_i$,
  $$\mathcal{S}_i(D_i) = \{(\overline{n}_i, 0)\}.$$
  If $\overline{n}_i > 3x_i$, then for every $D_i \in \mathcal{D}_i$,
  $$\mathcal{S}_i(D_i) = \{(r,s) \in \mathbb{Z}^2_+ : s = r - \underline{n}_i \text{ and } \underline{n}_i \le r \le \min\{|D_i| + \underline{n}_i, \overline{n}_i\}\},$$
  where $\underline{n}_i = 3x_i$.
- **Jiangsu, China policy:** The standard one-for-one policy is used with the restriction that the type of the blood supplied must be identical to the type of the patient (Chen, 2012): for every $D_i \in \mathcal{D}_i$, if $|\{d \in D_i : \beta_d = \beta_i\}| < \underline{n}_i$, then
  $$\mathcal{S}_i(D_i) = \{(0,0)\},$$
  and otherwise,
  $$\mathcal{S}_i(D_i) = \{(r,s) \in \mathbb{Z}^2_+ : s = r,\ \underline{n}_i \le r \le \min\{\overline{n}_i, |\{d \in D_i : \beta_d = \beta_i\}|\}\}.$$
  This is akin to no exchange (autarky) treatment. ∎

A **blood allocation problem with replacement donors** is denoted as $\mathcal{P} = \langle I, \beta_I, \overline{n}, D, \beta_D, v, \underline{n}, \mathcal{S} \rangle$. The inventory vector v, minimum guarantees $\underline{n}$, and feasible schedule correspondences $\mathcal{S}$ are interrelated and can all be considered as policy levers.[23] We fix every component of a problem except D.[24] Then a problem is simply denoted as a donor profile D.

Given a problem $D \in \mathcal{D}$, an **allocation** α consists of non-negative integers $\alpha_X(i)$ for each $i \in I$ and $X \in \mathcal{C}(\beta_i)$, and $\alpha(d) \in \{0,1\}$ for each $d \in \cup_{i \in I} D_i$ such that

1. for every $X \in \mathcal{B}$, $\sum_{i \in I : X \in \mathcal{C}(\beta_i)} \alpha_X(i) \le v_X + \sum_{d \in \cup_{i \in I} D_i : \beta_d = X} \alpha(d)$,

[22] Assume that x_i is exogenous to the problem, and the patient has not used the credits received from the previous donations in a replacement donor program.

[23] The vector v can be interpreted as the minimum required inventory level to be kept in stock. This is mostly ensured through a blood exchange program among blood banks, which is commonly practiced (for example, see AABB, 2020).

[24] Without loss of generality, we use this notation for brevity, assuming β_D is determined once D is given. Moreover, in Section 4.3, we discuss the effect of changing a patient's feasible schedule correspondence.

2. for every $i \in I$, $\big(\alpha(i), \sum_{d \in D_i} \alpha(d)\big) \in \mathcal{S}_i(D_i)$, where $\alpha(i) = \sum_{X \in \mathcal{C}(\beta_i)} \alpha_X(i)$.

In an allocation, the patients only receive blood that is medically compatible with them. An allocation specifies the amount of blood of each compatible type that a patient receives, as well as which of her donors donate. The first condition in the definition makes sure that, for each blood type, the allocated blood is not more than the sum of the existing blood in the blood bank and the collected blood from the patients' donors. Thus, it is a market clearing condition. The second condition requires that each patient's schedule induced by the allocation is in her feasible schedule set, which is determined by the set of donors that she brings forward. There always exists an allocation by definition. Denote the set of allocations for D as $\mathcal{A}(D)$.

We next introduce the patients' preferences. Each patient first and foremost cares about the amount of blood she receives and has monotonic preferences over the units of blood received.[25] Fixing the amount received, she would like fewer of her donors to donate. Formally, each patient $i \in I$ has a preference relation, denoted by $\mathbf{R}_i$, over the schedules in the set $\mathbb{W}_i = \{0, 1, \ldots, \overline{n}_i\} \times \{0, 1, \ldots, \max_{D_i \in \mathcal{D}_i} |D_i|\}$ such that for every $(r, s), (r', s') \in \mathbb{W}_i$,

$$(r, s) \, \mathbf{R}_i \, (r', s') \iff r > r' \text{ or } \big[\, r = r' \text{ and } s \leq s' \,\big].$$

Thus, a patient is indifferent between two schedules when her amounts of blood received and supplied are the same under both schedules. On the other hand, she strictly prefers one schedule to the other if she receives more blood under the first, or if she receives the same amount of blood under both but her donors donate more under the latter.[26] Let $\mathbf{I}_i$ and $\mathbf{P}_i$ denote the symmetric and asymmetric components of $\mathbf{R}_i$, respectively.

Let $\mathbb{W} = \times_{i \in I} \mathbb{W}_i$. Then $w = (r_i, s_i)_{i \in I} \in \mathbb{W}$ denotes a schedule profile, where $w_i = (r_i, s_i)$ for every $i \in I$. For any allocation $\alpha \in \mathcal{A}(D)$, let $w(\alpha)$ be the schedule profile induced by α. That is,

$$w(\alpha) = \Big(\alpha(i), \sum_{d \in D_i} \alpha(d)\Big)_{i \in I}.$$

Two allocations α and α' are **welfare equivalent** if $w_i(\alpha) \, \mathbf{I}_i \, w_i(\alpha')$ for every $i \in I$. An allocation $\alpha \in \mathcal{A}(D)$ is **efficient** if it is not Pareto dominated by another allocation, i.e., there is no allocation $\alpha' \in \mathcal{A}(D)$ such that $w_i(\alpha') \, \mathbf{R}_i \, w_i(\alpha)$ for every $i \in I$ and $w_j(\alpha') \, \mathbf{P}_j \, w_j(\alpha)$ for some $j \in I$.

A **mechanism** is a function f that maps each problem $D \in \mathcal{D}$ to an allocation

[25] Such monotonicity was motivated in Section 2.2.

[26] The reason for the assumption of lexicographic preferences is because the volume of blood received is of first-order importance for a patient relative to the volume of blood her donors need to supply. A planned surgery is likely to be cancelled if the required amount of blood is not fully met.

$f(D) \in \mathcal{A}(D)$. A mechanism f is **efficient** if for every $D \in \mathcal{D}$, $f(D)$ is efficient.

We consider the patients' incentives for bringing forward their donors. We introduce two notions of incentive compatibility, one weak and one strong, where the latter one coincides with strategy-proofness in our domain.

As alluded in Section 2.4, blood donation is not as costly as solid organ donation, leading to a much less invasive procedure and fast replenishment of blood. Therefore, as long as the patient does not receive less blood, providing more donors to the system may not be as undesirable for the patient. Based on this motivation, we first introduce a weaker incentive compatibility concept. Specifically, we require that concealing some of her donors never causes a patient to receive more blood.[27]

A mechanism f is **donor monotonic** if for any $D \in \mathcal{D}, i \in I$ and $D'_i \subseteq D_i$ we have

$$f(D)(i) \geq f(D'_i, D_{-i})(i).$$

Next, we introduce a stronger incentive compatibility concept. A mechanism f is **strongly donor monotonic** if for any $D \in \mathcal{D}, i \in I$ and $D'_i \subseteq D_i$ we have

$$w_i\big(f(D)\big)\ \mathbf{R}_i\ w_i\big(f(D'_i, D_{-i})\big).$$

That is, bringing forward any subset of her donors does not make the patient strictly better off.

We view these two notions as the weakest and strongest incentive requirements in this domain. Thus, understanding the implications of incentive compatibility at these two extremes is crucial. These two notions turn out to be equivalent if we have fixed exchange rates embedded in the feasible schedule correspondence profile, and thus in all similar problems that were previously considered in the literature.

4 Optimal Mechanisms

We seek general classes of mechanisms that guarantee efficiency and (strong) donor monotonicity. As a natural and direct way to define a mechanism, imagine that the mechanism designer has an underlying preference relation over all possible combinations of individual schedules, and for each problem, chooses an optimal allocation that maximizes his preferences.

Formally, the mechanism designer has a complete, transitive, and antisymmetric **aggregate preference relation** $\succeq$ over all schedule profiles in the set $\mathbb{W}$. The asymmetric

[27] We focus on a patient's incentive to hide her donors. While situations in which a patient exaggerates her donors, i.e., reports a larger set of donors than she actually has, are theoretically conceivable, this type of manipulation is often practically infeasible, since donor registration requires legally verifiable donor identification information.

component of $\succeq$ is denoted as $\succ$. A mechanism f is **induced** by the aggregate preference relation $\succeq$ if for every problem $D \in \mathcal{D}$,

$$f(D) \in \left\{\alpha \in \mathcal{A}(D) \; : \; w(\alpha) \succeq w(\alpha'), \; \forall \alpha' \in \mathcal{A}(D)\right\}.$$

We define two additional conditions on the mechanism designer's preferences. First, the aggregate preference relation $\succeq$ is **aligned with patients' preferences** if for every two schedule profiles w and w' such that $w_i \, \mathbf{R}_i \, w'_i$ for all $i \in I$, we have $w \succeq w'$. That is, if every patient weakly prefers w to w', then the mechanism designer also weakly prefers w to w'. Second, we say $w \in \mathbb{W}$ is a **basic schedule profile** if $w \in \{0,1\}^{2|I|}$, i.e., each element of the vector is either 0 or 1. In a basic schedule profile, there is a subset of patients who each receive a single unit of blood, and a subset of patients who each supply a single unit. The aggregate preference relation $\succeq$ is **(additively) responsive** to the mechanism designer's preferences over the basic schedule profiles if for every schedule profile $w \in \mathbb{W}$ and basic schedule profiles $w', w'' \in \{0,1\}^{2|I|}$ such that $w + w' \in \mathbb{W}$ and $w + w'' \in \mathbb{W}$,

$$w' \succ w'' \iff w + w' \succ w + w''.$$

If a mechanism f is induced by an underlying aggregate preference relation of the mechanism designer that is aligned with the patients' preferences and responsive, then we refer to f as an **optimal mechanism**.

Remark 1. *Every optimal mechanism is efficient.*

The above fact follows immediately from the preference alignment assumption. The general class of optimal mechanisms includes all mechanisms previously investigated in the literature within the framework of multi-unit exchange under compatibility-based preferences that we are aware of, as well as many new interesting ones. We will give several examples of such mechanisms that are practical and can be easily implemented by the blood bank.

We first provide an attractive subclass of optimal mechanisms that sequentially optimize various objectives of the blood bank. This class nests various special cases in the literature and may be of particular interest for practical use. Let $\{N_k\}_{k=1}^{\bar{k}}$, $\bar{k} \geq 2$, be a sequence of nonempty subsets of patients, which we refer to as **target sets**, and $\tau : \{1, \ldots, \bar{k}\} \to \{\max, \min\}$, where $\tau(1) = \max$, be a **target function** that designates for each subset N_k whether a maximization or minimization target will be achieved.

Maximization, denoted by $\tau(k) = \max$, means that the total amount of blood received by the patients in N_k is maximized given that the previous objectives are already satisfied.

Minimization, denoted by $\tau(k) = \min$, means that the total amount of blood supplied by the (donors of) patients in N_k is minimized given that the previous objectives are already satisfied.

The mechanism is defined through the following iterative procedure with respect to the target sets $\{N_k\}_{k=1}^{\bar{k}}$ and the target function τ.

Sequential Targeting Procedure:

Given a problem $D \in \mathcal{D}$, we construct a sequence of subsets of allocations $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \ldots \supseteq \mathcal{A}_{\bar{k}}$ in $\bar{k}$ steps after initializing $\mathcal{A}_0 = \mathcal{A}(D)$.

Step 1. Let $\mathcal{A}_1 \subseteq \mathcal{A}_0$ be the subset of allocations that maximize the amount of blood received by the patients in N_1, that is:

$$\mathcal{A}_1 = \arg\max_{\alpha \in \mathcal{A}_0} \sum_{i \in N_1} \alpha(i).$$

$$\vdots$$

Given that $\mathcal{A}_{k-1}$ is constructed in Step $k-1$, $k \geq 2$, Step k is defined as follows.

Step k. There are two possible cases.

- If $\tau(k) = \max$, let $\mathcal{A}_k \subseteq \mathcal{A}_{k-1}$ be the subset of allocations in $\mathcal{A}_{k-1}$ that maximize the amount of blood received by the patients in N_k, that is:
$$\mathcal{A}_k = \arg\max_{\alpha \in \mathcal{A}_{k-1}} \sum_{i \in N_k} \alpha(i).$$
- If $\tau(k) = \min$, let $\mathcal{A}_k \subseteq \mathcal{A}_{k-1}$ be the subset of allocations in $\mathcal{A}_{k-1}$ that minimize the amount of blood supplied by the patients in N_k, that is:
$$\mathcal{A}_k = \arg\min_{\alpha \in \mathcal{A}_{k-1}} \sum_{d \in \cup_{i \in N_k} D_i} \alpha(d).$$

The set of allocations $\mathcal{A}_{\bar{k}}$ is the outcome of the procedure.

Observe that $\mathcal{A}_{\bar{k}}$ may involve allocations that are not welfare equivalent. To have a well-defined mechanism that achieves our desiderata, we make two assumptions jointly on the target sets $\{N_k\}_{k=1}^{\bar{k}}$ and the target function τ:

1. $\bar{k} \geq 2|I|$ and the last $2|I|$ target sets, $\{N_k\}_{k=\bar{k}-2|I|+1}^{\bar{k}}$, are each singletons such that every patient $i \in I$ appears exactly twice as $N_k = N_\ell = \{i\}$ for some distinct $k, \ell \geq \bar{k}-2|I|+1$, with one target as $\tau(k) = \max$ and the other target as $\tau(\ell) = \min$.
2. For every $k \in \{2, \ldots, \bar{k}\}$, if $\tau(k) = \min$, then for any $i \in N_k$ there exists $k' < k$ such that $i \in N_{k'}$ and $\tau(k') = \max$. That is, if we are going to minimize the blood

supplied by a group of patients, then for each of those patients, we should have maximized the blood received by some group that includes her at an earlier step.

The first condition guarantees that the outcome allocations of the procedure are welfare equivalent: we use the last $2\left|I\right|$ targets as tie breakers among the patients, in case the previous targets lead to a multiplicity of allocations in terms of welfare levels. As the preferences of the patients are lexicographic in receiving more blood and then supplying less blood, the second condition will ensure the efficiency of sequential targeting.

A **sequential targeting mechanism** is defined through the above procedure with respect to a sequence of target sets $\{N_k\}_{k=1}^{\bar{k}}$ and a target function τ that satisfy the above two conditions: it chooses an allocation from the outcome set of the procedure, $\mathcal{A}_{\bar{k}}$, executed for each problem $D \in \mathcal{D}$.

A sequential targeting mechanism is induced by a lexicographic preference relation of the mechanism designer, such that given any two schedule profiles, he prefers the one in which the first target set receives more blood; when the amounts of blood received by the first target set are the same, he prefers the one in which the second target set receives more blood (supplies less blood) if the target is maximization (minimization), and so on.

Theorem 1. *Every sequential targeting mechanism is an optimal mechanism.*

Different target sets and target functions induce different sequential targeting mechanisms. In practice, since blood transfusion is one of the most common medical procedures, the patients requesting blood can be highly heterogenous. Target sets can be designed based on many observable patient characteristics, such as disease type, medical urgency, blood type, previous donation record, and various demographic characteristics.

For example, the blood bank may want to design the mechanism such that the first target set includes all the women who will give birth, the second target set includes all the patients of blood type $O-$, and the third target set includes all other patients. The maximization target is assigned to these three sets, while the fourth target set includes all patients with a minimization target. The rationale behind this design is that severe bleeding is one of the most common complications that leads to maternal death.[28] In addition, it can be difficult to satisfy the need of patients of blood type $O-$, as they can only receive $O-$ blood which is also the universal donor under ABO-compatible and Rh D-compatible transfusion.

Notably, the class of sequential targeting mechanisms includes two important special cases, priority mechanisms and maximal mechanisms with priority tie-breakers, which

[28]According to Bates et al. (2008), in Sub-Saharan Africa, where the blood supply heavily depends on replacement donor systems, 26% of haemorrhage maternal deaths were due to lack of blood.

were examined by Manjunath and Westkamp (2021) and Andersson et al. (2021), respectively, in similar setups. In our context, these two classes of mechanisms are also more broadly defined due to the general specification of feasible schedules.

In a **priority mechanism**, the patients are processed one at a time using a priority order. Let $|I| = n$ and list the patients in this order as $i_1, i_2, \ldots, i_n$: the mechanism first maximizes the welfare of i_1; then, among all allocations that achieve this goal, it maximizes the welfare of i_2, and so on. Formally, the target sets are singletons such that $N_{2k-1} = N_{2k} = \{i_k\}$ for every $k \in \{1, \ldots, n\}$. The target function τ is defined as $\tau(2k-1) = \max$ and $\tau(2k) = \min$ for every $k \in \{1, \ldots, n\}$.[29]

In a **maximal mechanism with priority tie-breakers**, the total amount of blood received by all the patients is maximized, then the total amount of blood donated by all replacement donors is minimized. List the patients as $i_1, i_2, \ldots, i_n$ using a priority tie-breaker. Then among all total welfare maximizing allocations, the welfare of i_1 is maximized. Subject to this goal being satisfied, the welfare of i_2 is maximized, and so on. Formally, the first two target sets are the set of all patients: $N_1 = N_2 = I$. The remaining target sets are singletons such that $N_{2k-1} = N_{2k} = \{i_{k-1}\}$ for every $k \in \{2, \ldots, n+1\}$. The target function τ is defined as $\tau(2k-1) = \max$ and $\tau(2k) = \min$ for every $k \in \{1, \ldots, n+1\}$.[30]

Another interesting class of optimal mechanisms are **weighted maximal mechanisms**. Instead of having an aggregate preference relation that is lexicographic over the targets, the mechanism designer may assign weights to different targets, and maximize the difference between a weighted sum of the blood received by different groups of patients and a weighted sum of the blood supplied by different groups of patients. Such weights assigned to the targets lead to individual weights for each patient. Thus, in general, the designer can have a linear score function defined on the schedule profiles such that for every schedule profile $w = (r_i, s_i)_{i \in I} \in \mathbb{W}$,

$$O(w) = \sum_{i \in I} \Big(W^r(i) r_i - W^s(i) s_i \Big),$$

where $\big(W^r(i), W^s(i)\big) \in \mathbb{R}^2_+$ are the individual weights assigned to each patient $i \in I$. Let $I = \{1, \ldots, |I|\}$. A weighted maximal mechanism with respect to the score function O is a mechanism that is induced by the aggregate preference relation $\succeq$, defined as follows.

[29]The priority mechanisms are counterparts of the serial dictatorships that are widely studied in the context of object allocation with strict preferences.

[30]Maximal mechanisms have found wide-spread application in the context of kidney exchange, which involves single-unit demand for each patient. For example, in the US, the UNOS National Kidney Exchange Program and Alliance for Paired Donation have adopted maximal mechanisms, although they use different tie-breakers than the patient-based priority approach (Sönmez and Ünver, 2017).

For any two schedule profiles w and w' such that $w \neq w'$, let $w \succeq w'$ if $O(w) > O(w')$, or, $O(w) = O(w')$ and there exists $k \in \{1, \ldots, |I|\}$ such that $w_k \ \mathbf{P}_k \ w'_k$ and $w_\ell \ \mathbf{I}_\ell \ w'_\ell$ for all $\ell < k$. In addition, let $w \succeq w$ for any schedule profile w. It is straightforward to check that $\succeq$ is complete, transitive, antisymmetric, and responsive. Moreover, to ensure that it is aligned with the patients' preferences, we assume that for every $i \in I$ and $D_i \in \mathcal{D}_i$, $W^r(i) \geq W^s(i)|D_i|$.[31] Then, a weighted maximal mechanism is an optimal mechanism. Moreover, the class of weighted maximal mechanisms subsumes the sequential targeting mechanisms (see Appendix C.1 in Supplemental Material).[32]

4.1 Donor Monotonicity

In this subsection, we explore the incentives faced by patients in bringing forward their full sets of donors to the blood bank.

For a general profile of feasible schedule correspondences $\mathcal{S}$, the optimal mechanisms may not be incentive compatible even in the donor monotonicity sense. We will state regularity conditions on the feasible schedule correspondences that many real-life policies—such as one-for-one exchange—obey.

We make three assumptions which ensure that the optimal mechanisms are donor monotonic. They all have natural explanations. The first one is about the convexity of a feasible schedule set for a given set of donors. Generally, a set $S \subseteq \mathbb{Z}^2_+$ is **L-convex** (where L stands for lattice) if for every $x, y \in S$, we have

$$\left\lfloor \frac{x+y}{2} \right\rfloor, \left\lceil \frac{x+y}{2} \right\rceil \in S.$$

L-convexity is one of the two most used generalizations of convexity to discrete domains.[33]

Assumption 1 (L-convexity)**.** *The feasible schedule set $\mathcal{S}_i(D_i)$ is L-convex for every $i \in I$ and $D_i \in \mathcal{D}_i$.*

Figure 1 provides a geometric illustration with three examples of L-convex feasible schedule sets. Assumption 1 also guarantees that an outcome allocation of a weighted maximal mechanism can be found in polynomial time, as shown in Appendix C.2 in Supplemental Material.

[31] This assumption implies that for any $w, w' \in \mathbb{W}$ such that $w_i \ \mathbf{R}_i \ w'_i$ for all $i \in I$, we have $O(w) \geq O(w')$.

[32] It is also worth mentioning that given a general optimal mechanism induced by an aggregate preference relation $\succeq$, there may not exist a linear utility function that represents $\succeq$, and thus the class of optimal mechanisms is strictly larger than the class of weighted maximal mechanisms.

[33] The other one is *M-convexity*, where M stands for matroid. See Murota (2013) for a general treatment of discrete convexity notions and discrete convex analysis.

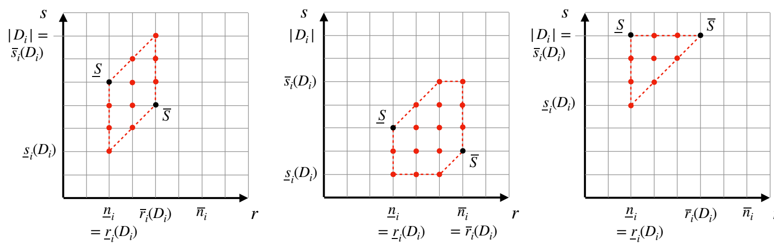

Figure 1: Illustration of Assumption 1, L-convexity. The feasible schedule set $\mathcal{S}_i(D_i)$ is the integral points of a convex polygon with integral corners and at most six edges of slopes 1, 0, or ∞. It is a lattice with the minimum schedule marked as $(\underline{r}_i(D_i), \underline{s}_i(D_i))$ and the maximum schedule marked as $(\overline{r}_i(D_i), \overline{s}_i(D_i))$. Observe that by the definition of feasible schedule correspondences $|D_i| \geq \overline{s}_i(D_i)$ and $\overline{n}_i \geq \overline{r}_i(D_i)$. The best schedule $\overline{S}$ and the worst schedule $\underline{S}$ are also marked in each graph to show the lexicographic orientation of the patient's preferences in more blood received first and less blood supplied second.

A special case that satisfies Assumption 1 is the classical one-for-one exchange rate between the blood received and supplied, as depicted in Figure 2.

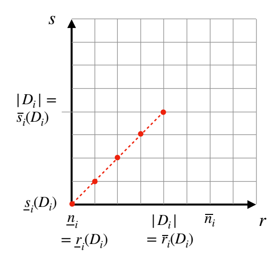

Figure 2: An L-convex feasible schedule set induced by the one-for-one exchange rate policy. In this example, we assume $\overline{n}_i > |D_i|$; if $|D_i| \geq \overline{n}_i$, then $\overline{r}_i(D_i) = \overline{n}_i$ is the maximum amount of blood that can be received.

The second assumption generalizes the idea that each unit of blood has a positive "price." It says that when a patient receives more (or less) blood, there is a feasible schedule in which her donors also donate more (or less) blood. Note that the patient does not have to supply more blood when she receives more, but this assumption says that such a schedule is feasible.

Assumption 2 (Feasibility of positive price)**.** *For every patient $i \in I$ and donor set $D_i \in \mathcal{D}_i$, the feasible schedule set $\mathcal{S}_i(D_i)$ satisfies the following:*

- *if $(r, s), (r', s') \in \mathcal{S}_i(D_i)$, $r' > r$ and $s < |D_i|$, then there exists $s'' > s$ such that $(r', s'') \in \mathcal{S}_i(D_i)$; and*

- *if* $(r, s), (r', s') \in \mathcal{S}_i(D_i)$*,* $r' < r$ *and* $s > 0$*, then there exists* $s'' < s$ *such that* $(r', s'') \in \mathcal{S}_i(D_i)$*.*

That is, given a feasible schedule, if the patient can potentially receive a larger (or smaller) amount of blood, then she can potentially receive this amount by supplying more (or less), as long as the supply does not exceed her number of donors (or is non-negative). The one-for-one exchange rate policy satisfies the feasibility of positive price assumption: each additional unit received costs exactly one unit supplied.

L-convexity and feasibility of positive price are independent. For example, the two-for-one exchange rate policy, i.e., two units supplied for each unit received, satisfies feasibility of positive price but not L-convexity;[34] the second feasible schedule set in Figure 1 violates feasibility of positive price as it has a "flat top" at $s = \overline{s}_i(D_i) < |D_i|$ and a "flat bottom" at $s = \underline{s}_i(D_i) > 0$, while it is L-convex. The other sets in this figure satisfy feasibility of positive price, although the third one has a "flat top." This is because the "flat top" occurs at the maximum possible supply $s = |D_i|$.

Before presenting the final assumption, we introduce a concept regarding the ranking of schedule sets for the patients, which will also be useful in the comparative static analysis in Section 4.3. Given a patient $i \in I$, a donor set $D_i \in \mathcal{D}_i$ and two sets $S, S' \subseteq \mathbb{W}_i$, we say S is **weakly more favorable than** S' **at** D_i if the following holds:

- if $(r, s) \in S'$ and $r \geq \underline{n}_i$, then there exists $s' \leq s$ such that $(r, s') \in S$; and
- if $(r, s) \in S$, $s \leq |D_i|$ and $(r, s') \in S'$, then there exists $s'' \geq s$ such that $(r, s'') \in S'$.

When S and S' are schedule sets for a patient, S is weakly more favorable than S' at her donor set if (i) for any schedule in S' such that the amount received is at least the minimum guarantee, there is a schedule in S where the patient receives the same amount by supplying weakly less blood, and (ii) for any schedule in S such that the amount supplied does not exceed the number of donors, whenever there is a schedule in S' where she receives the same amount of blood, there is a schedule in S' where she receives this amount by supplying weakly more blood.

Using this concept, we make the following assumption regarding the relation between feasible schedule sets when a patient reports different sets of donors.

Assumption 3 (Non-diminishing favorability in donors)**.** *For every patient* $i \in I$ *and donor sets* $D_i, D'_i \in \mathcal{D}_i$ *such that* $D'_i \subseteq D_i$*,* $\mathcal{S}_i(D_i)$ *is weakly more favorable than* $\mathcal{S}_i(D'_i)$ *at* D'_i*.*

[34]See Section 5.1.3 for a detailed discussion of this policy.

Favorability manifests itself geometrically as $\mathcal{S}_i(D_i)$ being an expansion of $\mathcal{S}_i(D_i')$ in the direction of receiving more blood, and/or a downward shift of $\mathcal{S}_i(D_i')$.[35] In addition to Assumptions 1 and 2, the one-for-one exchange rate policy satisfies non-diminishing favorability in donors as well, since the feasible schedule set simply expands when the number of donors increases. In Figures 3 and 4, we give two examples involving endogenously determined exchange rates to further illustrate the implications of Assumption 3 in conjunction with Assumptions 1 and 2.

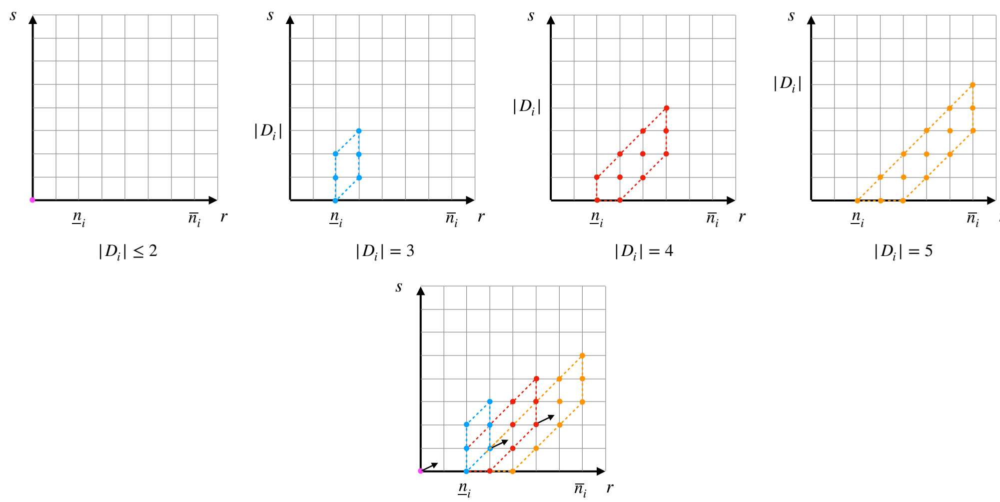

Figure 3: An illustration of a feasible schedule correspondence $\mathcal{S}_i$ satisfying Assumptions 1, 2, and 3. This particular policy relies only on the number of donors brought forward $|D_i|$ but not other specifics of the donor set. The minimum guarantee of patient i is $\underline{n}_i = 2$ while her maximum need is $\overline{n}_i = 7$. The top graphs illustrate $\mathcal{S}_i(D_i)$ for $|D_i| = 0, \ldots, 5$, while the bottom graph shows how the feasible schedule set changes as the number of donors increases.

The main result of this section is as follows:

Theorem 2. *Under Assumptions 1, 2, and 3, every optimal mechanism is donor monotonic.*[36]

The proof of this result is substantially involved and we relegate it to Appendix A. We give a sketch of the proof here.

[35] The comparison of feasible schedule sets based on favorability is in similar spirit to the weak set order in Che, Kim, and Kojima (2019), which is used in establishing weak monotone comparative statics in games with strategic complementarities and other models, although our comparative static exercises in Section 4.3 are not related to theirs.

[36] We actually prove a stronger version of this theorem: If the three assumptions are imposed on a patient $i \in I$ and only Assumption 1 (L-convexity) is imposed on the other patients, then patient i cannot receive more units of blood by under-reporting her donor set.

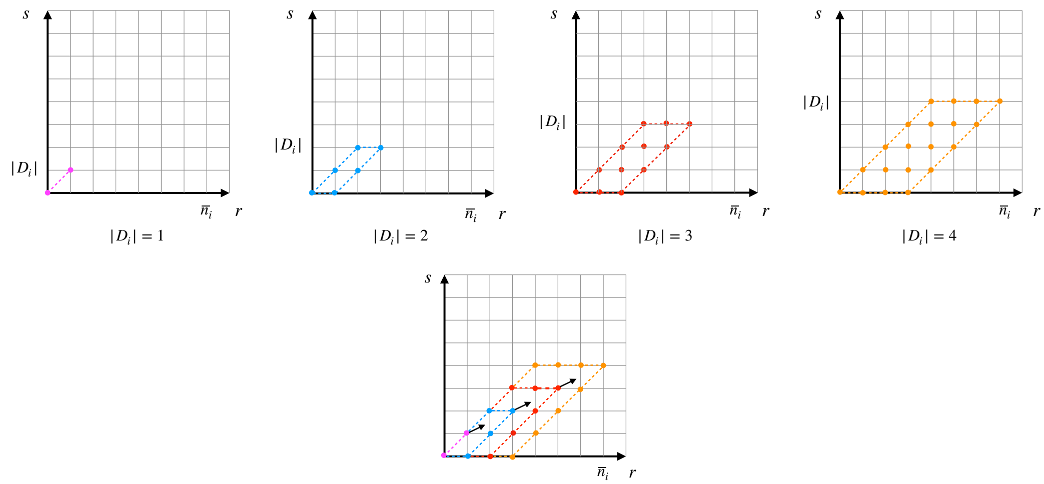

Figure 4: An illustration of a feasible schedule correspondence $\mathcal{S}_i$ satisfying Assumptions 1, 2, and 3. This particular policy relies only on the number of donors $|D_i|$ but not other specifics of the donor set. The minimum guarantee of patient i is $\underline{n}_i = 0$ while her maximum need is $\overline{n}_i = 7$. The top graphs illustrate $\mathcal{S}_i(D_i)$ for $|D_i| = 1, \ldots, 4$. The bottom graph shows how the feasible schedule set changes as the number of donors increases.

Proof Sketch. We first define an auxiliary matching market that is isomorphic to the original problem, which we refer to as an *extended problem*. In this market, the blood bank is represented as a pseudo-patient and its inventory is represented by pseudo-donors paired with it. For each blood type, we also introduce a dummy patient paired with dummy donors so that, without loss of generality, we can focus on the simple case where any patient cannot receive blood from her own compatible donors. In such an extended problem, a *matching* specifies which donors are matched with each patient. A patient is not only matched with the donors who donate to her, but also those of her donors who do not donate to anyone. Hence, this is a pure exchange economy. The analogue of a mechanism for extended problems is a *rule*, which assigns a matching to each extended problem. We then define *optimal rules*, which are isomorphic to the optimal mechanisms. An optimal rule chooses a matching by maximizing a strict and responsive preference relation of the mechanism designer over the extended schedule profiles (that also include the dummy patients' schedules). In Lemma 2, we show that for each optimal mechanism, there is an optimal rule that is welfare equivalent to it for the real patients. Hence, to prove the theorem, it is sufficient to show that every optimal rule is donor monotonic. The rest of the proof consists of two lemmata.

The first one, Lemma 3, is the most crucial result in the proof. This lemma essentially gives a general necessary condition for profitable manipulation under any rule. Consider two extended problems: the original one, denoted as $\hat{D}$, and the one induced by some patient i concealing exactly one of her donors, denoted as $\hat{D}'$. Let M be a matching for $\hat{D}$ and M' be a matching for $\hat{D}'$ such that i receives more blood under M'. Then, Lemma 3 says that there exists a particular graph theoretical structure, a *cycle* or a *chain*, relating these two matchings. A cycle C from the matching M to the matching M' is a list of patients and donors in which each patient i points to a donor that is matched with i under M' but not under M, and each donor d points to the patient that is matched with d under M. We can "add" the cycle C to the matching M to make it closer to M': starting from M, we remove each donor d in the cycle from the match of the patient that is pointed by d, and add it to the match of the patient that points to d. Due to Assumptions 1, 2, and 3, the definition of a cycle is carefully tailored to ensure that these exchanges lead to a well-defined matching for $\hat{D}$, denoted as $M + C$. We can also "remove" the cycle from M': starting from M', we remove each donor d in the cycle from the match of the patient that points to d, and add it to the match of the patient that is pointed by d. This results in a matching for $\hat{D}'$, denoted as $M' - C$. On the other hand, a chain is similar to a cycle. The only differences are that the head patient in the chain does not point to any donor, and the tail patient in the chain is not pointed by any donor. Chain addition and removal operations are similarly defined and also lead to new matchings for the two extended problems.

Finally, Lemma 4 states that the optimal rules are donor monotonic. We proceed by contradiction. Let F be an optimal rule, $\hat{D}$ be an extended problem, and $\hat{D}'$ be the extended problem induced by patient i concealing a donor. Suppose that patient i receives more blood under the matching $F(\hat{D}')$ than under the matching $F(\hat{D})$. By Lemma 3, there is a cycle or a chain C from $F(\hat{D})$ to $F(\hat{D}')$. Then, $F(\hat{D}) + C$ is a matching for $\hat{D}$ and $F(\hat{D}') - C$ is a matching for $\hat{D}'$. If $F(\hat{D})$ and $F(\hat{D}) + C$ are not welfare equivalent, then the mechanism designer must strictly prefer the schedule induced by $F(\hat{D})$ to the schedule induced by $F(\hat{D}) + C$. In the cycle or chain operations, the amount of blood received or supplied by any patient is adjusted by at most one unit. Hence we can use the fact that the mechanism designer's preferences are responsive to the preferences over basic schedules to show that the schedule induced by $F(\hat{D}') - C$ is strictly preferred to the schedule induced by $F(\hat{D}')$, which is a contradiction. Therefore, $F(\hat{D})$ and $F(\hat{D}) + C$ are welfare equivalent. Then, as patient i still receives more blood under $F(\hat{D}')$ than under $F(\hat{D}) + C$, we can apply Lemma 3 again to show that there is a cycle or a chain C' from $F(\hat{D}) + C$ to $F(\hat{D}')$. By similar arguments as before, $F(\hat{D}) + C$ and $(F(\hat{D}) + C) + C'$ are welfare equivalent. Hence $F(\hat{D})$ and $(F(\hat{D}) + C) + C'$ are welfare equivalent. This process can be continued infinitely, which leads to a contradiction since each cycle or chain addition generates a new matching that is closer to $F(\hat{D}')$. ∎

Each of the three assumptions is needed for the donor monotonicity of the optimal mechanisms. In Appendix D in Supplemental Material, Example S.3 shows that Assumption 1 is necessary. In this example, Assumption 1 is violated while Assumptions 2 and 3 are satisfied, and a priority mechanism is not donor monotonic. Similarly, in Example S.4 in the same appendix, only Assumption 2 is violated, and a priority mechanism is not donor monotonic. Finally, it is straightforward to show that Assumption 3 is necessary. For example, for every patient i, if she brings no donors, then she receives her minimum guarantee of $\underline{n}_i = 1$ unit of blood: her feasible schedule set is $\{(1,0)\}$; if she brings forward any donor, then her feasible schedule set shrinks to $\{(0,0)\}$. Such a feasible schedule correspondence violates Assumption 3, but satisfies Assumptions 1 and 2. In this case, any mechanism is manipulable including an optimal mechanism.

4.2 Strong Donor Monotonicity

In order for the optimal mechanisms to be strongly donor monotonic, we need a stronger restriction on the relation between feasible schedule sets when a patient reports different donor sets.

Assumption 4. *For every patient $i \in I$ and donor sets $D_i, D_i' \in \mathcal{D}_i$ such that $D_i' \subseteq D_i$, we have*

- *if $(r,s) \in \mathcal{S}_i(D_i')$ and $r \geq \underline{n}_i$, then there exists s' such that $(r,s') \in \mathcal{S}_i(D_i)$,*
- *if $(r,s) \in \mathcal{S}_i(D_i)$ and $(r,s') \in \mathcal{S}_i(D_i')$, then $s \leq s'$.*

It is straightforward to see that Assumption 4 implies Assumption 3. Therefore, under Assumptions 1, 2 and 4, the optimal mechanisms are donor monotonic. Moreover, in this case, if a patient reports a subset of her donors and still receives the same amount of blood, then the second condition in Assumption 4 implies that her donors do not donate less blood. Hence, we have the following result.

Theorem 3. *Under Assumptions 1, 2, and 4, every optimal mechanism is strongly donor monotonic.*

One important circumstance under which strong donor monotonicity can be achieved is when the feasible schedule correspondences feature exogenous exchange rates, in the sense that for every possible amount of blood received in a feasible schedule set, there is a unique amount of supply associated with it. That is, for every $i \in I$, $D_i \in \mathcal{D}_i$

and $(r, s) \in \mathcal{S}_i(D_i)$, there does not exist $s' \neq s$ such that $(r, s') \in \mathcal{S}_i(D_i)$. In this case, Assumption 3 and Assumption 4 are equivalent.

Remark 2. *Suppose that the exchange rates are exogenous. Then Assumptions 1, 2, and 3 pin down a particular class of feasible schedule correspondences. Assume that for every $i \in I$, $D_i \neq \emptyset$ for some $D_i \in \mathcal{D}_i$, then Assumptions 1, 2, and 3 are satisfied if and only if the following is true for every $i \in I$:*

- *for every $D_i \in \mathcal{D}_i$ such that $\mathcal{S}_i(D_i) \neq \{(0,0)\}$, there exist $\underline{s}_i(D_i), \overline{r}_i(D_i) \in \mathbb{Z}_+$, where $\underline{s}_i(D_i) \leq |D_i|$, $\underline{s}_i(D_i) = 0$ if $\underline{n}_i = 0$, and $\underline{n}_i \leq \overline{r}_i(D_i) \leq \overline{n}_i$, such that*
$$\mathcal{S}_i(D_i) = \{(r, s) \in \mathbb{W}_i : s - \underline{s}_i(D_i) = r - \underline{n}_i, s \leq |D_i|, \text{ and } \underline{n}_i \leq r \leq \overline{r}_i(D_i)\},$$
- *for every $D_i \in \mathcal{D}_i$ and $D_i' \subseteq D_i$ such that $\mathcal{S}_i(D_i) \neq \{(0,0)\}$ and $\mathcal{S}_i(D_i') \neq \{(0,0)\}$, $\underline{s}_i(D_i) \leq \underline{s}_i(D_i')$ and $\overline{r}_i(D_i) \geq \overline{r}_i(D_i')$, and*
- *for every $D_i \in \mathcal{D}_i$ and $D_i' \subseteq D_i$, $\mathcal{S}_i(D_i) = \{(0,0)\}$ implies $\mathcal{S}_i(D_i') = \{(0,0)\}$.*

*Thus, if a patient i participates in the program, then she has to supply $\underline{s}_i(D_i)$ units to receive her minimum guarantee. Beyond this schedule, she has to supply one additional unit for each additional unit received, with the maximal amount received being restricted by $\overline{r}_i(D_i)$. We refer to such feasible schedule correspondences as **two-part tariffs**, which include both the one-for-one exchange rate policy and the Xi'an policy in Example 1 as special cases. We give another example of a two-part tariff in Figure 5.*

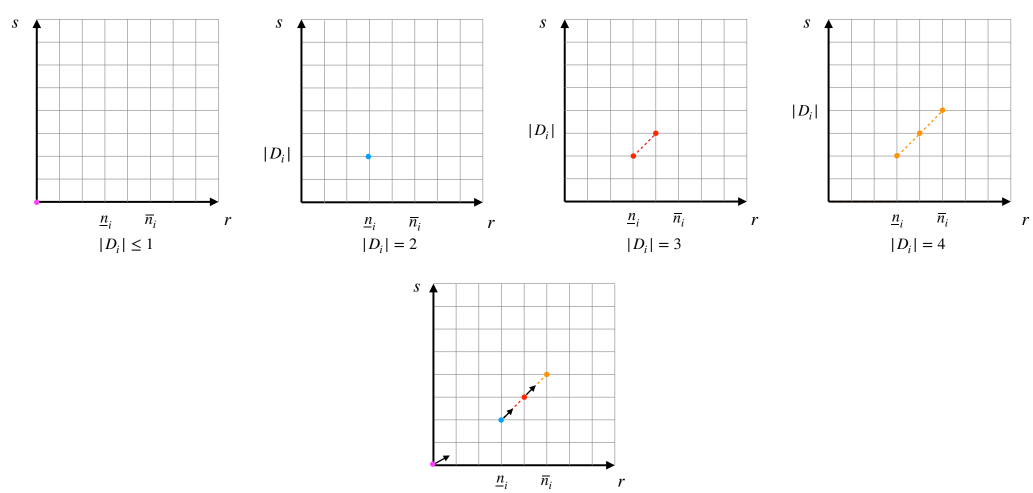

Figure 5: An illustration of the two-part tariff policy. The patient i has to supply two units to receive her minimum guarantee of $\underline{n}_i = 3$ units. The top graphs illustrate $\mathcal{S}_i(D_i)$ for $|D_i| \in \{0, \ldots, 4\}$, while the bottom graph shows how the feasible schedule set changes as the number of donors increases.

Strong donor monotonicity of optimal mechanisms can also be achieved under feasible schedule correspondences that incorporate endogenous exchange rates. An example is given in Figure 6.

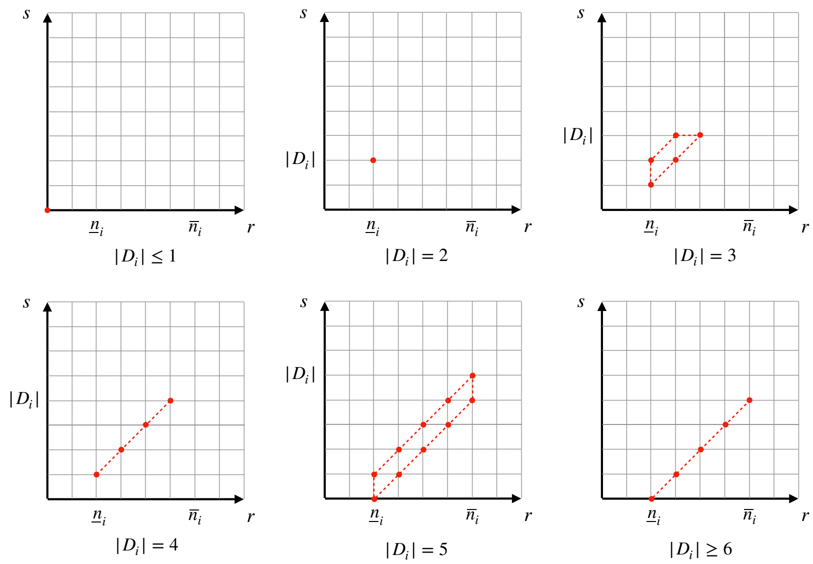

Figure 6: An illustration of a feasible schedule correspondence satisfying Assumptions 1, 2, and 4. Exchange rates are endogenous when the patient i has three or five donors.

4.3 Comparative Statics

In establishing the donor monotonicity of the optimal mechanisms, we need Assumption 3, which requires that if a patient i brings forward a donor set D_i larger than D_i', i.e., $D_i' \subseteq D_i$, then $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}_i(D_i')$ at D_i'. A weakly more favorable feasible schedule set is given to the patient to incentivize her to report the full set of donors. It is then natural to consider the effect of making her feasible schedule set weakly more favorable, while keeping her donor set fixed. To this end, we introduce a notation to denote the possibility of changing the underlying feasible schedule correspondences. For a given profile of feasible schedule correspondences $\mathcal{S} = (\mathcal{S}_i)_{i \in I}$ and an optimal mechanism f, let $f\big(D \,|\, \mathcal{S}\big)$ be the outcome of f for any $D \in \mathcal{D}$ under $\mathcal{S}$.

Theorem 4. *Suppose that Assumptions 1 and 2 are satisfied for all feasible schedule correspondences considered. Consider any optimal mechanism f, any $D \in \mathcal{D}$, and any patient $i \in I$. If $\mathcal{S}$ and $\mathcal{S}'$ are two profiles of feasible schedule correspondences such that $\mathcal{S}_j(D_j) = \mathcal{S}_j'(D_j)$ for all $j \in I \setminus \{i\}$, and $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}_i'(D_i)$ at D_i, then*

$$w_i\Big(f\big(D \,|\, \mathcal{S}\big)\Big) \, \mathbf{R}_i \, w_i\Big(f\big(D \,|\, \mathcal{S}'\big)\Big).$$

That is, if a patient is given a weakly more favorable feasible schedule set, then she is weakly better off under the same optimal mechanism, when Assumptions 1 and 2 are satisfied. The proof of this result is similar to that of Theorem 2 with certain modifications.

5 Policy Discussion and Simulations

In this section, we first discuss how certain practical challenges in designing blood allocation policies can be addressed using our framework. Next, we explain practical implementation details regarding possible day-to-day functioning of a blood bank that adopts our solutions. We conclude the section with simulations that show the possible gains from our proposal.

5.1 Policy Design with Feasible Schedule Sets

Our framework has two main tools that can be used to satisfy different objectives in blood allocation. One involves the adoption of practical optimal mechanisms such as the sequential targeting mechanisms, which was discussed in Section 4. On the other hand, the feasible schedule correspondences can be designed to impose exchange rate policies and achieve more nuanced objectives regarding fairness, efficiency, and incentives, which we discuss in this subsection. In addition, we extend the baseline model to discuss policy design regarding blood bank inventory objectives (by treating the blood bank as an *agent* in optimal mechanism design), such as maximizing certain blood type inventory, and integrated blood component markets in Appendix E in Supplemental Material.

5.1.1 Equitable Blood Allocation

An important flexibility of our proposal is that the exchange rates can be determined endogenously. This can be especially useful when some patients may potentially have few or no paired donor candidates. We can design policies that accommodate for patients with and without donors as equitably as possible, keeping incentive and efficiency properties of the optimal mechanisms intact.

An example is provided in Figure 7. In this example, the feasible schedule correspondence of patient i satisfies Assumptions 1, 2, and 3. She can receive the minimum guarantee of $\underline{n}_i = 1$ unit of blood even if she does not have a donor. She can also receive up to her maximum need of $\overline{n}_i = 3$ units in this case. As she brings forward more donors, her chances of receiving more units of blood beyond $\underline{n}_i = 1$ weakly increase by donor monotonicity. Moreover, under such a policy her donors never donate more blood than what she receives.

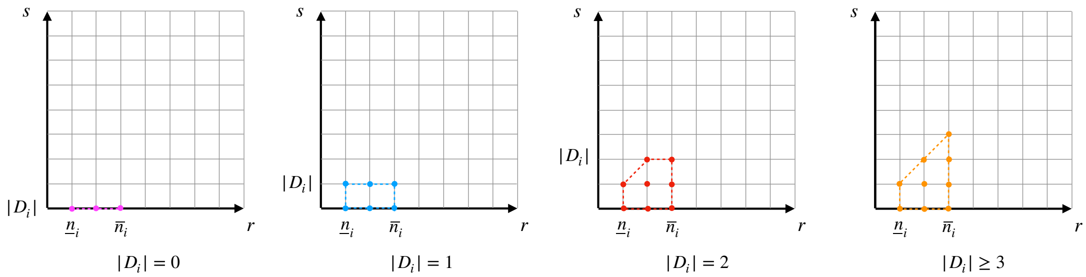

Figure 7: An equitable feasible schedule policy.

Our proposal is also compatible with some existing equitable replacement donor policies. For example, in leading Chinese hospitals, patients whose hometowns are away from the city where the hospital is located are often not required to supply as many donors as those local patients. The rationale behind this policy is that relatives of patients from other cities are usually not readily available to donate on behalf of the patients. Similarly, in Cambodia, replacement donor requirements are waived for a patient if she has no next-of-kin (Davies, 2004). Thus, the patient-specific nature of the feasible schedule correspondences can accommodate such fairness considerations as well.

In addition to fairness, a flexible policy with endogenous exchange rates can also help address some ethical concerns about replacement donor programs and enhance the overall efficiency of the system. Under a fixed exchange rate policy that is commonly observed around the world, a patient without enough donors may be forced to recruit illegal professional donors, leading to the issue of black markets for blood. On the other hand, if a fixed exchange rate, such as the one-for-one rate, is strictly enforced, then a patient without any donor cannot receive any blood even if the blood bank does have enough inventory for her, leading to obvious welfare loss.

In general, given the fairness, efficiency, and ethical issues of a fixed exchange rate policy, although rules may be bent in some way in practice, our design formalizes flexible and endogenous exchange rates, bringing rigor and transparency to the allocation system.

5.1.2 Blood Type Targeting

Blood banks occasionally fall short in blood components of certain blood types while others are aplenty. For example, the blood type distribution varies across different regions of the world, but AB Rh D$-$ is almost always the rarest type and components of this type will be in short supply. On the other hand, although ABO-identical transfusion is required for certain blood components in some countries, this compatibility requirement is often relaxed in other cases. For instance, under ABO-compatible red blood cell transfusion,

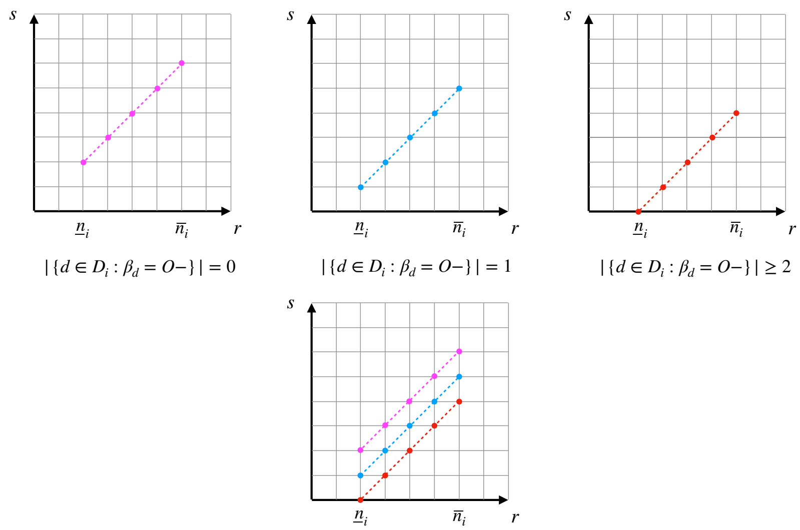

Figure 8: A feasible schedule policy that provides stronger incentives to reveal type $O-$ donors. In each case, the feasible schedule set of patient i consists of those schedules on the graph in which the amount supplied does not exceed her number of donors, and is $\{(0,0)\}$ if there is no such schedule. This feasible schedule correspondence satisfies Assumptions 1, 2, and 4. Assume that these assumptions are also satisfied for the other patients. Then, any optimal mechanism is strongly donor monotonic. Furthermore, if patient i has one or two type $O-$ donors, concealing a type $O-$ donor leads to a strictly worse outcome for her under any optimal mechanism.

blood type O Rh D− is the universal donor, and under ABO-plasma compatible platelet transfusion, blood type AB is the universal donor. Therefore, it may be important for a blood bank to target its collection of certain types of blood. Since a patient's feasible schedule set depends on the observable characteristics of her donor set, this goal can be achieved by incentivizing the provision of donors of desired blood types through feasible schedule policies. In Figure 8, we provide an example of a schedule design that favors bringing forward more type O Rh D− donors. In this case, a patient is able to receive the same amount of blood by supplying less if she has donors of blood type O Rh D−.

5.1.3 Beyond One-for-One Exchange

As mentioned before, some countries in Africa (e.g., Congo and Cameroon) and Mexico, for various reasons, use two-for-one exchange rate: two units of blood need to be supplied for each unit received. However, feasible schedule correspondences accommodating the two-for-one exchange rate violate Assumption 1. Example S.3 in Appendix D in Supplemental Material shows that a priority mechanism may not be donor monotonic under such an exogenous exchange rate policy, due to L-convexity not being satisfied.

However, we can generate endogenous exchange rate policies that closely approximate the two-for-one exchange rate, such that under these policies the optimal mechanisms are donor monotonic. See Figure 9 for an example. Such an approach can also be applied to approximate other exogenous exchange rates.

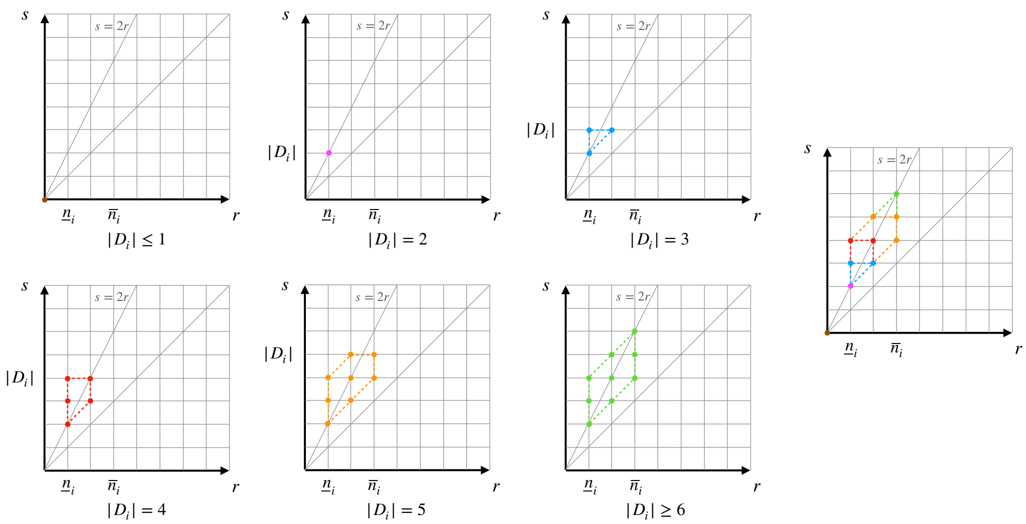

Figure 9: A feasible schedule correspondence is designed to approximate the two-for-one exchange rate. The patient i is required to supply two units to receive her minimum guarantee. For any $s \in \{2, \ldots, 6\}$ such that $s \leq |D_i|$, $(\frac{1}{2}s, s)$ should be a feasible schedule when s is an even number, and we consider $(\lfloor \frac{s}{2} \rfloor, s)$ and $(\lceil \frac{s}{2} \rceil, s)$ feasible schedules when s is an odd number. Then the above graphs illustrate the feasible schedule correspondence that assigns the smallest set of schedules that include these feasible schedules in each case so that Assumptions 1, 2, and 3 are satisfied for patient i.

5.2 Practical Implementation

Unlike solid organ exchanges, replacement donor programs do not require the simultaneity of donation and transfusion. As mentioned earlier, this gives us flexibility to schedule donations and transfusions separately. Moreover, the donated blood must be tested and processed for safety reasons, which makes it unsuitable for immediate transfusion. It may take up to 24 hours to test and process donated blood. Thus, replacement donor programs are usually operationalized for non-urgent care patients and blood banks have to function through slack inventories to help urgent care patients. We envision that our proposed replacement donor programs will be complementary to the system for urgent care patients like those in the current practice. Moreover, we believe the dynamic feature of this problem in which patients arrive over time is less crucial in implementation once initial conditions are set.[37]

We propose the establishment of a donor registry system that allows patients to register information about their potential replacement donors at the time they are seeking blood. A potential donor registered into the system may later be utilized depending on her blood type or the amount of blood the patient will end up receiving. When a certain threshold of potential donors is reached (for example, this could be daily for logistical reasons),[38] one of our practical optimal mechanisms is implemented to determine the actual blood assignment of non-urgent patients together with the potential replacement donors that are requested to donate blood. After the chosen donors donate and the blood is tested and processed, the medical procedures requiring transfusions will be conducted in the following days, or if the slack is large in the blood bank, then the replacement donor blood can be used to replenish the inventory after the patients receive blood in the preceding days.[39] If the patients receive transfusions before their donors donate, then reneging can be an issue. However, issues of reneging on donation promises are reported to be insignificant according to at least one center we have been in touch with.[40]

5.3 Simulations

In this subsection, we report the results of simulations comparing an optimal mechanism under two different feasible schedule correspondence profiles and an emulation of current practices. We envision the scenario that we simulate as follows. A blood bank starts the day with an inventory of blood. Then, during the day, a number of patients arrive sequentially. The blood types of each patient, replacement donor, and unit in the blood bank inventory are drawn randomly and independently using the Indian blood-type distribution in Table 1. We simulate red blood cell transfusion using packs that include some amount of donor plasma and thus follow the commonly practiced ABO-identical

[37]There have been several studies recently on organ exchanges which showed that dynamic concerns can be of secondary order under different assumptions, starting with Ünver (2010). In our view, the dynamic features may be important to optimize initial conditions, such as daily inventory reallocation across several blood banks or interval between consecutive optimizations, which we envision to be daily initially. In the daily functioning of a replacement donor program, however, once these initial conditions are fixed, we believe the implementation details we present here address most practical concerns, if not all. These two issues can indeed be arenas of future investigations.

[38]The optimal threshold would naturally depend on other factors, such as the arrival rate of patients and donors, and the distribution of blood types in the population. This is beyond the scope of the current model.

[39]Unfortunately, the blood bank inventory slack is usually very small in many places such as India.

[40]Based on personal communication with the director of the Tucuman Blood Bank, Dr. Felicitas Agote, on July 7, 2020. Moreover, even for costly kidney donation, only six donors reneged out of more than 1700 transplants in the US in chain donations where patients receive a kidney donation before their donors donate (Cowan et al., 2017).

$O+$	$A+$	$B+$	$AB+$	$O-$	$A-$	$B-$	$AB-$
27.85%	20.80%	38.14%	8.93%	1.43%	0.57%	1.79%	0.49%

Table 1: Blood-type frequencies in India (RhesusNegative.net, 2012-2019).

and Rh D-compatible protocol. We consider three patient set sizes $|I| = 25, 50, 100$, representing medium to large hospital systems and their blood banks. Each patient i is assumed to need a maximum of $\overline{n}_i$ units, determined by an independent and identical draw from the uniform distribution with the support set $\{1, 2, \ldots, 6\}$. The minimum guarantee is set to zero for all patients. Each patient i has a donor set D_i such that $|D_i|$ is determined by an independent and identical draw from the uniform distribution with the support set $\{0, 1, \ldots, 5\}$, so that in expectation her number of donors is one less than her maximum need. The number of units in the inventory of the blood bank is determined uniformly from the support set $\Big\{0, 1, \ldots, \big\langle 5\rho|I|\big\rangle\Big\}$, where $\rho \in \Big\{0, \frac{1}{50}, \frac{1}{25}, \frac{1}{10}, \frac{1}{5}, \frac{1}{2}, 1\Big\}$, 5 is the maximum donor number of each patient, and $\big\langle x\big\rangle$ rounds x to the nearest integer. Therefore, ρ is the ratio of the maximum number of units in the blood bank inventory to the maximum number of replacement donors, which we use as a policy parameter. We consider the following three allocation protocols.

1. **First-Come First-Serve**: We emulate the current practices in replacement donor programs using a simple protocol in which each patient receives as much blood as possible up to her maximum need upon arrival, either from her own donors or the blood bank inventory. For each unit she receives from the blood bank, one of her randomly determined donors donates to the blood bank, implementing a one-for-one exchange rate. Then, with the updated inventory of the blood bank, the next arriving patient is served similarly and so on.
2. **One-for-one Optimal**: We use the sequential targeting mechanism that maximizes the total units received by the patients under the feasible schedule correspondences with the one-for-one fixed exchange rate.
3. **Flexible Optimal**: We use the same sequential targeting mechanism as the previous one under the feasible schedule correspondences in which each patient is asked to supply at most one unit more than the amount she receives and at least one unit less than the amount received.[41]

[41] Note that in this case Assumptions 1, 2, and 3 are satisfied, and hence the mechanism is donor monotonic.

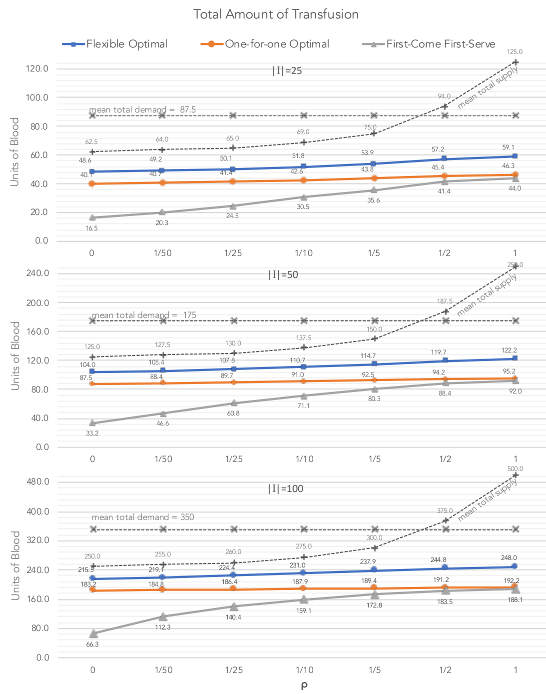

Figure 10: Total units of blood transfused to the patients in the simulations for patient set sizes $|I| = 25,\ 50,\ 100$ as a function of ρ (the ratio of the maximum units in the blood bank inventory to the maximum number of replacement donors), under the three allocation protocols.

5.3.1 Simulation Results

We randomly generate 1,000 markets and summarize their average results through two figures. Figure 10 displays the average total transfusion under each allocation protocol. Figure 11 displays the cumulative distribution of transfusions among patients (the top panels) and the marginal distribution of net demand calculated as the difference between units received and units supplied under flexible optimal (the bottom panels),[42] for two cases with $|I| = 50$: when there is no blood bank inventory and when the maximum blood bank inventory is one-tenth of the maximum number of replacement donors. We chose these two particular parameter sets to summarize the more detailed results. The first one is an extreme case, while the other represents a more realistic inventory level.

One immediate finding from Figure 10 is that the flexible optimal protocol always leads to more transfusion than the one-for-one optimal protocol, which in turn leads to more transfusion than the first-come first-serve protocol. For $|I| = 50$, flexible optimal

[42]For other protocols, due to the one-for-one exchange rate, the net demand is always 0.

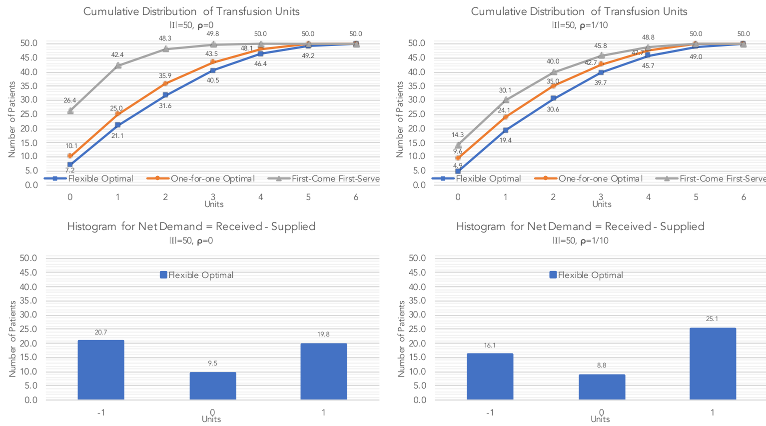

Figure 11: Distributions of transfusion units (top panels) and net demand (bottom panels) in the simulations for $|I| = 50$, when $\rho = 0$ (left panels) and $\rho = \frac{1}{10}$ (right panels).

leads to 19%-28% more transfusion than one-for-one optimal, and the percentage gain is monotonic in increasing blood bank inventory. On the other hand, one-for-one optimal leads to 164% to 3% more transfusion than first-come first-serve for $|I| = 50$. However, in this case, the percentage gain is monotonic in decreasing blood bank inventory. First-come first-serve is very inefficient, especially when the blood bank inventory is very small. For $|I| = 50$ and $\rho = 0$, only 33 units of blood are transfused (Figure 10), with more than 26 patients receiving no blood and no patient receiving more than 4 units (Figure 11, top left panel). First-come first-serve overcomes its inefficiency to some degree as the blood bank inventory increases. When we have equal blood supply from the blood bank and from the replacement donors, it becomes very close to one-for-one optimal for all population sizes. Moreover, one-for-one optimal is the least sensitive to the blood bank inventory levels among all allocation protocols. Flexible optimal takes advantage of the variable exchange rates and responds well to the increase in inventory, and it is still more effective than the others for small inventories.

Flexible optimal first-order stochastically dominates one-for-one optimal, and one-for-one optimal first-order stochastically dominates first-come first-serve in transfusion unit distributions (Figure 11, top panels), which hold for most other simulation parameters as well. Flexible optimal helps 86%-90% of patients with at least one unit of transfusion. When $\rho = 0$, the number of oversupplying patients and the number of undersupplying patients under this protocol are close to each other, and less than one fifth of the patients neither oversupply nor undersupply (Figure 11, bottom left panel). As the blood bank supply increases, at least half of the patients undersupply (Figure 11, bottom right panel).

6 Related Literature and Concluding Remarks

The literature on market design for living-donor kidney exchange spanned by Roth, Sönmez, and Ünver (2004, 2005, 2007) in economics is related to the current paper, although most of this literature is about exchanging one transplant organ for one donor's organ with the following notable exception. The complementarities in the initial blood units supplied are similar to the complementarities in dual organ exchanges in Ergin, Sönmez, and Ünver (2017). However, the one-for-one exchange rate is not crucial in our model while it is important in the latter study. The differences in institutional details between solid organ exchange applications and our main application are explained in Section 2. Our two donor monotonicity notions would reduce to the donor monotonicity notion introduced in Roth, Sönmez, and Ünver (2005) if patients had unit demand and the exchange rate were one-for-one.

The WHO guidelines suggest that blood should only come from VNRDs and economic incentives can adversely affect both blood safety and blood donation. The position of the WHO has been questioned based on recent evidence (Lacetera, Macis, and Slonim, 2013). In particular, Lacetera, Macis, and Slonim (2012) provide evidence from a natural field experiment showing that economic incentives have a positive effect on voluntary donation and can encourage pro-social behavior. Additionally, Slonim, Wang, and Garbarino (2014) also study blood donation from an economic perspective, and discuss methods to increase blood supply and improve the supply and demand balance without market prices. Pay-it-forward and pay-it-backward incentive schemes for encouraging COVID-19 convalescent plasma donation have recently been proposed by Kominers et al. (2020) in a market design context.[43]

There are not many papers on mechanism or market design for multi-unit exchange of indivisible goods, even under the restriction of one-for-one exchange rate. Besides Ergin, Sönmez, and Ünver (2017), two notable exceptions are Manjunath and Westkamp (2021), who study shift exchanges for medical doctors and other professionals as a market design problem,[44] and Andersson et al. (2021), who consider the design of time banks

[43]They propose issuing vouchers for the convalescent plasma donation of patients who recover from COVID-19 that can be used by these donors' family members who may become sick in the future to gain prioritized access to plasma therapy. They also propose issuing vouchers to patients who pledge to donate after recovery in return for their own prioritized access to plasma therapy. Since one donor can donate plasma that can treat more than one patient, with an appropriate number of vouchers and willingness to donate, the system can collect enough plasma to treat all patients. Their paper inspects the steady-state analysis of a stylized large-market model, while ours is on mechanism design in a finite environment.

[44]In Manjunath and Westkamp (2021), for each agent there are three indifference classes of objects:

or favor barter markets to be cleared by centralized clearinghouses.[45] Our paper as well as Andersson et al. (2021) substantially generalizes the priority mechanism introduced for bilateral kidney exchange, i.e., one-for-one donor exchange between two patients with unit demand, by Roth, Sönmez, and Ünver (2005).[46,47]

Price discovery and Pareto efficient allocation through endogenously determined exchange rates are the main features of competitive equilibrium. For the allocation of indivisible goods, this approach was first studied by Hylland and Zeckhauser (1979) using pseudo-market equilibrium from equal "fake" monetary incomes and extended to settings with multi-unit demand and deterministic outcomes by using an approximate equilibrium concept by Budish (2011). This approach generally fails to guarantee the existence of a competitive equilibrium with endowments and no monetary income—as in our model—even with single-unit demand under dichotomous preferences and the possibility of probabilistic assignments (see Garg, Tröbst, and Vazirani, 2020 for an impossibility and Echenique, Miralles, and Zhang, 2020 on how to obtain possibility results with single-unit demand in related domains with some fake money injection). Moreover, competitive equilibrium as a mechanism is not incentive compatible in small markets.

Similar to our main insight in the blood allocation context, Agarwal et al. (2019) underline and calculate the welfare loss in the US kidney exchange due to inefficient mechanisms and agency problems. They argue that while the number of transplants

desirable objects, undesirable objects that she is endowed with, and undesirable objects that she is not endowed with. This trichotomous preference domain is more general than ours, and suits their application of shift exchange but not our blood allocation problem. They consider priority mechanisms that are greedy in the sense of serial dictatorships and show that they are individually rational, efficient, and strategy-proof. This class of mechanisms is a special case of our optimal mechanisms with one-for-one exchange rate.

[45] As in our study, Andersson et al. (2021) also consider dichotomous preferences, but their domain is more restrictive, since each agent is endowed with identical copies of an object. Compared to Manjunath and Westkamp (2021), they are able to achieve the stronger efficiency requirement of maximality with a less general preference domain. They study maximal mechanisms with priority tie-breakers and show that they are individually rational and strategy-proof. This class of mechanisms is a special case of our optimal mechanisms with one-for-one exchange rate, and our results in the general model subsume this paper's results (see Appendix B in Supplemental Material).

[46] Matching models with unit demand and compatibility-based dichotomous preferences have been studied in the context of graph theory—for example, see Lovász and Plummer (1986) for an excellent survey of this discrete mathematics literature. The incentive and fairness properties of mechanisms on such graphs were first analyzed by Bogomolnaia and Moulin (2004) in an economic model of a two-sided matching market. A recent related paper regarding matching and assignment with dichotomous preferences is Nicolò, Sen, and Yadav (2019), who study the assignment of tasks to pairs of agents where each agent has separable dichotomous preferences over her assigned partner and task. This paper focuses on finding core matchings in this domain.

[47] Another recent paper on the multi-unit exchange model with one-for-one exchange rate, Aziz (2019), derives a sufficient condition for the strategy-proofness of a mechanism.

that can be performed crucially depends on the marginal product of each patient-donor pair, current platform rules largely ignore this variation in the social value of submissions, much like the inefficiency caused by fixed exchange rates in blood allocation.

Our work is also related to the literature on manipulation incentives via misreporting endowment information.[48] We add to this literature by showing possibility results in a compatibility-based preferences model.

In closing, it is our hope that in addition to developing the theory for efficient blood allocation mechanisms with good incentive properties, our approach will be an important first step toward a blueprint of transparent, equitable, and systematic replacement donor programs that are in line with the goals of the WHO. Relaxing the constraints imposed by fixed exchange rates, this approach can help to overcome important practical frictions such as coercion and emerging black markets in places where these programs are not adequately organized.

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[48]This literature starts with Postlewaite (1979). In classical exchange economies, he shows that all individually rational and efficient rules are manipulable via hiding endowments, while there exists an individually rational and efficient rule that is immune to manipulation via destroying endowments. In this classical domain, agents can consume their hidden endowments in addition to what they acquire by exchange. However, in our domain this is not possible: blood needs to be processed before transfusion, and thus no patient can use her hidden donors' blood. Atlamaz and Klaus (2007) consider a multi-object assignment setting and show that individually rational and efficient rules are generally manipulable via hiding or destroying endowments. Sertel and Özkal-Sanver (2002) study manipulation via endowments in the two-sided matching market. In the context of airline landing slot assignment, Schummer and Abizada (2017) show that while any efficient rule is manipulable via slot destruction, a positive result emerges under a weaker form of efficiency suitable for that context.

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Online Appendix

A Proofs

A.1 Proof of Theorem 1

Theorem 1 follows from the fact that every sequential targeting mechanism is a weighted maximal mechanism, which is proved in Appendix C.1 in Supplemental Material.

A.2 Proof of Theorem 2

We first show that Assumptions 1, 2 and 3 imply the following two assumptions on the feasible schedule correspondences, which will be useful in our proof.

Assumption 1′. For every $i \in I$, $D_i \in \mathcal{D}_i$, and $(r,s),(r',s') \in \mathcal{S}_i(D_i)$, we have

1. If $r' > r$ and $s' > s$, then
$$(r+1,s+1) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r'-1,s'-1) \in \mathcal{S}_i(D_i).$$
2. If $r' > r$ and $s' \leq s$, then
$$(r+1,s) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r'-1,s') \in \mathcal{S}_i(D_i).$$
3. If $s' > s$ and $r' \leq r$, then
$$(r,s+1) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r',s'-1) \in \mathcal{S}_i(D_i).$$

Assumption 2′. For every $i \in I$, $D_i, D_i' \in \mathcal{D}_i$ with $D_i' \subseteq D_i$, $(r,s) \in \mathcal{S}_i(D_i)$ and $(r',s') \in \mathcal{S}_i(D_i')$, we have

1. If $r' > r, s' > 0$ and $s < |D_i|$, then
$$(r+1,s+1) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r'-1,s'-1) \in \mathcal{S}_i(D_i').$$
2. If $r' > r$ and $s' \leq s$, then
$$(r+1,s) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r'-1,s') \in \mathcal{S}_i(D_i').$$

Lemma 1. *Assumption 1′ and Assumption 2′ are satisfied.*

Proof of Lemma 1. Consider any $i \in I$ and $D_i \in \mathcal{D}_i$. Let $\mathcal{S}_i(D_i) = S$. For any $x, y \in \mathbb{W}_i$, where $x = (r,s)$ and $y = (r',s')$, denote $\text{dis}(x,y) = r' - r + s' - s$, and $y > x$ if $r' > r$ and $s' > s$. Suppose that $x = (r,s) \in S$, $y = (r',s') \in S$, and $y > x$. We want to first show that $(r+1,s+1) \in S$. If $\text{dis}(x,y) = 2$, then we are done. If $\text{dis}(x,y) > 2$, then consider $z = \left\lceil \frac{x+y}{2} \right\rceil > x$. By Assumption 1 (L-convexity),

$z \in S$. It follows from $\text{dis}(x,y) > 2$ that either $\lceil \frac{r+r'}{2} \rceil < r'$ or $\lceil \frac{s+s'}{2} \rceil < s'$. Hence $2 \leq \text{dis}(x,z) < \text{dis}(x,y)$. If $\text{dis}(x,z) > 2$, we can repeat the argument and find $z' \in S$ such that $z' > x$ and $2 \leq \text{dis}(x,z') < \text{dis}(x,z)$. Continuing in this fashion, in the end we must have $(r+1, s+1) \in S$. By symmetric arguments, it can be shown that $(r'-1, s'-1) \in S$. So Condition 1 in Assumption 1′ is satisfied.

Next we show Condition 2. Suppose that $x = (r,s) \in S$, $y = (r', s') \in S$, $r' > r$ and $s' \leq s$. First, we argue that there exists $s'' \leq s$ such that $(r+1, s'') \in S$. If $r' = r+1$, we are done. If $r' > r+1$, then consider $\lceil \frac{x+y}{2} \rceil = (r_1, s_1)$. We have $r' > r_1 > r$ and $s_1 \leq s$. By Assumption 1, $(r_1, s_1) \in S$. If $r_1 > r+1$, we can repeat the argument and find $(r_2, s_2) \in S$ such that $r_1 > r_2 > r$ and $s_2 \leq s$. Therefore, eventually we have $(r+1, s'') \in S$ for some $s'' \leq s$. Denote $z = (r+1, s'')$. If $s'' < s$, consider $\lceil \frac{x+z}{2} \rceil = (r+1, s_3)$. Then $s'' < s_3 \leq s$. By Assumption 1, $(r+1, s_3) \in S$. If $s_3 < s$, we can repeat the argument and find some s_4 such that $(r+1, s_4) \in S$ and $s_3 < s_4 \leq s$. Therefore, we must have $(r+1, s) \in S$. By symmetric arguments, it can be shown that $(r'-1, s') \in S$. Finally, Condition 3 in Assumption 1′ can be shown in a similar way as the proof of Condition 2.

To show Assumption 2′, consider any $i \in I$, $D_i, D'_i \in \mathcal{D}_i$ with $D'_i \subseteq D_i$, $(r,s) \in \mathcal{S}_i(D_i)$ and $(r', s') \in \mathcal{S}_i(D'_i)$.

Suppose that $r' > r, s' > 0$ and $s < |D_i|$. Since $r' > 0$, by the definition of feasible schedule correspondences, $\mathcal{S}_i(D'_i) \neq \{(0,0)\}$ and $r' \geq \underline{n}_i$. Then by Assumption 3 (Non-diminishing favorability in donors), there exists s_1 such that $(r', s_1) \in \mathcal{S}_i(D_i)$. Since $r' > r$ and $s < |D_i|$, by Assumption 2 (Feasibility of positive price), there exists $s_2 > s$ such that $(r', s_2) \in \mathcal{S}_i(D_i)$. Then, given that $(r', s_2) > (r, s)$, it follows from Condition 1 in Assumption 1′ that $(r+1, s+1) \in \mathcal{S}_i(D_i)$. This also implies that $\mathcal{S}_i(D_i) \neq \{(0,0)\}$, and hence $r \geq \underline{n}_i$. Recall that $\mathcal{S}_i(D'_i) \neq \{(0,0)\}$. So there exists s_3 such that $(\underline{n}_i, s_3) \in \mathcal{S}_i(D'_i)$. Since $r' > r \geq \underline{n}_i$ and $s' > 0$, by Assumption 2, there exists $s_4 < s'$ such that $(\underline{n}_i, s_4) \in \mathcal{S}_i(D'_i)$. Then, given that $(r', s') > (\underline{n}_i, s_4)$, it follows from Condition 1 in Assumption 1′ that $(r'-1, s'-1) \in \mathcal{S}_i(D'_i)$.

It remains to show Condition 2 in Assumption 2′. Suppose that $r' > r$ and $s' \leq s$. Then $r' \geq \underline{n}_i$. By Assumption 3, there exists $s_1 \leq s' \leq s$ such that $(r', s_1) \in \mathcal{S}_i(D_i)$. It follows from Condition 2 in Assumption 1′ that $(r+1, s) \in \mathcal{S}_i(D_i)$. Then, we argue that $(r, s') \in \mathcal{S}_i(D_i)$. This is true if $s' = s$. Suppose that $s' < s$. Then consider $(r', s_1) \in \mathcal{S}_i(D_i)$ and $(r, s) \in \mathcal{S}_i(D_i)$, where $r' > r$ and $s_1 \leq s' < s$. By repeated applications of Condition 3 in Assumption 1′, we have $(r, s') \in \mathcal{S}_i(D_i)$. Finally, since $\mathcal{S}_i(D_i) \neq \{(0,0)\}$, $r \geq \underline{n}_i$. Given that $r' > r \geq \underline{n}_i$, $(r', s') \in \mathcal{S}_i(D'_i)$ and $(\underline{n}_i, s_2) \in \mathcal{S}_i(D'_i)$ for some s_2, it is straightforward to see that, by Assumption 1, there exists s_3 such that

$(r, s_3) \in \mathcal{S}_i(D_i')$. Since $(r, s') \in \mathcal{S}_i(D_i)$ and $s' \leq |D_i'|$, by Assumption 3, there exists $s_4 \geq s'$ such that $(r, s_4) \in \mathcal{S}_i(D_i')$. As $(r', s') \in \mathcal{S}_i(D_i')$, $r' > r$ and $s' \leq s_4$, it follows from Condition 2 in Assumption 1′ that $(r' - 1, s') \in \mathcal{S}_i(D_i')$. ■

We introduce new machinery to prove this theorem. In particular, we will construct *extended problems* in which each blood type has a replica and there are some new dummy agents. Such a construction mainly serves two purposes: it helps us represent allocations as *matchings*, which specify the donors that each patient receives blood from; it allows us to focus on the simple case where no patient receives blood from her own (compatible) donors.

First, we treat the blood bank b as if it were a *pseudo patient* and introduce a donor set for it. It has a set of (volunteer non-remunerated) donors D_b, where for each blood type $X \in \mathcal{B}$ the blood bank has v_X donors. That is, for each $X \in \mathcal{B}$, $\big|\{d \in D_b : \beta_d = X\}\big| = v_X$.

Then, for each blood type $X \in \mathcal{B}$, we construct a *dummy blood type* $\hat{X}$. Let $\hat{\mathcal{B}} = \mathcal{B} \cup \{\hat{X} : X \in \mathcal{B}\}$. Define $\hat{\mathcal{C}}(\cdot)$ as follows: for each $X \in \mathcal{B}$,

$$\hat{\mathcal{C}}(X) = \mathcal{C}(X) \cup \{\hat{Y} : Y \in \mathcal{C}(X)\} \qquad \text{and} \qquad \hat{\mathcal{C}}(\hat{X}) = \{X\}.$$

For each $X \in \mathcal{B}$, we construct a *dummy patient* $i_{\hat{X}}$ and her set of *dummy donors* $D_{i_{\hat{X}}}$, such that

$$\beta_{i_{\hat{X}}} = \beta_d = \hat{X} \quad \text{for every } d \in D_{i_{\hat{X}}},$$
$$\overline{n}_{i_{\hat{X}}} = \big|D_{i_{\hat{X}}}\big| = \sum_{i \in I} \overline{n}_i, \text{ and}$$
$$\underline{n}_{i_{\hat{X}}} = 0.$$

Moreover, let her feasible schedule set be

$$\mathcal{S}_{i_{\hat{X}}}(D_{i_{\hat{X}}}) = \big\{(r, s) \ : \ 0 \leq r \leq \overline{n}_{i_{\hat{X}}} \quad \text{and} \quad s = r\big\}.$$

For any problem $\mathcal{P} = \langle I, \beta_I, \overline{n}, D, \beta_D, v, \underline{n}, \mathcal{S}\rangle$, under $\mathcal{B}$ and $\mathcal{C}(\cdot)$, which has been simply denoted as $D = (D_i)_{i \in I}$, we use $\hat{D} = \big(D, D_b, (D_{i_{\hat{X}}})_{X \in \mathcal{B}}\big)$ to denote the **extended problem**, under $\hat{\mathcal{B}}$ and $\hat{\mathcal{C}}(\cdot)$, after we add the blood bank as a pseudo patient, its inventory as a donor set, the dummy patients, and the dummy donors to the problem $\mathcal{P}$. Note that in an extended problem, everything is fixed except the donor sets of the real patients I.

Given an extended problem $\hat{D}$, let $\hat{I} = I \cup \{b\} \cup \{i_{\hat{X}}\}_{X \in \mathcal{B}}$ and $\hat{\mathbf{D}} = \cup_{i \in \hat{I}} D_i$. From now on in this proof, we refer to each $i \in \hat{I}$ as a **patient** (in reality it can be a real patient, a dummy patient, or the blood bank) and each $d \in \hat{\mathbf{D}}$ as a **donor** (it can be a real donor, a dummy donor, or a unit of blood in the blood bank's inventory). A(n) **(auxiliary) matching** is a function $M : \hat{I} \to 2^{\hat{\mathbf{D}}}$, where the **match** of every patient $i \in \hat{I}$, $M(i)$, is denoted as M_i by a slight abuse of notation, such that

1. $M_i \cap M_j = \emptyset$ for every $i, j \in \hat{I}$ with $i \neq j$, and $\cup_{i \in \hat{I}} M_i = \hat{\mathbf{D}}$,
2. for every $i \in \hat{I} \setminus \{b\}$ and $d \in M_i \setminus D_i$, $\beta_d \in \hat{\mathcal{C}}(\beta_i)$,
3. for every $i \in \hat{I} \setminus \{b\}$, $(|M_i \setminus D_i|, |D_i \setminus M_i|) \in \mathcal{S}_i(D_i)$.

Let $\mathcal{M}(\hat{D})$ be the set of all the matchings for $\hat{D}$. Every allocation $\alpha \in \mathcal{A}(D)$ in the problem D is associated with a matching $M \in \mathcal{M}(\hat{D})$ in its extended problem $\hat{D}$ and vice versa, as we show in a claim in the proof of Lemma 2 below. In particular, the match of a patient $i \in \hat{I} \setminus \{b\}$ consists of two parts.

- The first part $M_i \setminus D_i$ is the set of donors that she receives blood from. These donors necessarily belong to other patients, and the blood types of these donors are compatible with patient i (Condition 2 in the definition of a matching).
- The second part $M_i \cap D_i$ is the set of her own donors who end up not donating. They are matched back with patient i.[49]

Therefore, patient i never receives blood from her own donors in a matching. Although this may not be the case in an allocation, we introduced the dummy patients and their dummy donors to account for this possibility. If in an allocation a patient $i \in I$ receives blood from one of her own donors, this is represented in a matching in the following manner:[50]

- this donor $d \in D_i$ is matched with the dummy patient induced by her blood type, $i_{\hat{\beta}_d}$,
- patient i is matched with one of the dummy donors of this dummy patient, i.e., with some $d' \in D_j$ where $j = i_{\hat{\beta}_d}$, in return.

As a result, the set of donors of any patient $i \in \hat{I} \setminus \{b\}$ who actually donate in a matching M is $D_i \setminus M_i$. Therefore, $(|M_i \setminus D_i|, |D_i \setminus M_i|)$ has to be in the feasible schedule set $\mathcal{S}_i(D_i)$ (Condition 3 in the definition of a matching). Two matchings M and M' are **welfare equivalent** if $|M_i \setminus D_i| = |M'_i \setminus D_i|$ and $|D_i \setminus M_i| = |D_i \setminus M'_i|$ for every $i \in \hat{I} \setminus \{b\}$.

The analogue of a mechanism in the extended problems is a **rule**, which is a function F that maps each extended problem $\hat{D}$ to a matching $F(\hat{D}) \in \mathcal{M}(\hat{D})$. A rule F is **donor monotonic** if for any $D, D' \in \mathcal{D}$ and $i \in I$ such that $D'_i \subseteq D_i$ and $D'_j = D_j$ for every $j \in I \setminus \{i\}$, we have

$$|F_i(\hat{D}) \setminus D_i| \geq |F_i(\hat{D}') \setminus D'_i|.$$[51]

We define optimal rules for the extended problems by extending the optimal mechanisms. For each $X \in \mathcal{B}$, let $\mathbb{W}_{i_{\hat{X}}} = \{0, 1, \ldots, \overline{n}_{i_{\hat{X}}}\}^2$. A vector $\hat{w} = (r_i, s_i)_{i \in \hat{I} \setminus \{b\}} \in$

[49]Similarly, the blood bank b receives donations from the donors $M_b \setminus D_b$, while keeping the donors $M_b \cap D_b$.

[50]See the proof of Lemma 2 for the details of this construction.

[51]Note that we do not consider manipulations by the dummy patients as their donor sets are fixed.

$\times_{i\in\hat{I}\setminus\{b\}}\mathbb{W}_i$ denotes an **extended schedule profile**. For each extended problem $\hat{D}$ and matching $M \in \mathcal{M}(\hat{D})$, let $\hat{w}(M)$ be the extended schedule profile induced by M. That is,

$$\hat{w}(M) = \big(\big|M_i \setminus D_i\big|, \big|D_i \setminus M_i\big|\big)_{i\in\hat{I}\setminus\{b\}}.$$

Suppose that $\hat{\succeq}$ is a complete, transitive and antisymmetric preference relation over the extended schedule profiles, with $\hat{\succ}$ denoting its asymmetric component. Moreover, $\hat{\succeq}$ is **responsive**: for every $\hat{w}, \hat{w}', \hat{w}'' \in \times_{i\in\hat{I}\setminus\{b\}}\mathbb{W}_i$ such that $\hat{w}', \hat{w}'' \in \{0,1\}^{2(|\hat{I}|-1)}$ and $\hat{w}+\hat{w}', \hat{w}+\hat{w}'' \in \times_{i\in\hat{I}\setminus\{b\}}\mathbb{W}_i$,

$$\hat{w}' \mathrel{\hat{\succ}} \hat{w}'' \iff \hat{w}+\hat{w}' \mathrel{\hat{\succ}} \hat{w}+\hat{w}''.$$

Consider a rule F. If for each extended problem $\hat{D}$,

$$F(\hat{D}) \in \{M \in \mathcal{M}(\hat{D}) : \hat{w}(M) \mathrel{\hat{\succeq}} \hat{w}(M'), \forall M' \in \mathcal{M}(\hat{D})\},$$

then we say F is an **optimal rule**, **induced** by $\hat{\succeq}$.[52]

Given any problem $D \in \mathcal{D}$, we say an allocation $\alpha \in \mathcal{A}(D)$ and a matching $M \in \mathcal{M}(\hat{D})$ are **welfare equivalent** if for every $i \in I$, $\alpha(i) = \big|M_i \setminus D_i\big|$ and $\sum_{d\in D_i}\alpha(d) = \big|D_i \setminus M_i\big|$. The following result implies that to prove Theorem 2, it is sufficient to show that every optimal rule is donor monotonic, under Assumptions 1, 2 and 3.

Lemma 2. *For every optimal mechanism f, there is an optimal rule F such that for every $D \in \mathcal{D}$, $f(D)$ and $F(\hat{D})$ are welfare equivalent.*

Proof of Lemma 2. We first prove the following claim:

Claim. Consider any $D \in \mathcal{D}$. For every $\alpha \in \mathcal{A}(D)$, there is $M \in \mathcal{M}(\hat{D})$ such that α and M are welfare equivalent. For every $M \in \mathcal{M}(\hat{D})$, there is $\alpha \in \mathcal{A}(D)$ such that M and α are welfare equivalent.

Proof. Let $D \in \mathcal{D}$. We prove the claim in two parts.

Part 1. Let $\alpha \in \mathcal{A}(D)$. Consider the extended problem $\hat{D}$, and any blood type $X \in \mathcal{B}$. Since $\big|D_{i_{\hat{X}}}\big| = \sum_{j\in I}\overline{n}_j$, there exists a collection of disjoint donor sets $\big\{M_j^{\hat{X}}\big\}_{j\in I:X\in\mathcal{C}(\beta_j)}$ such that for every $j \in I$ with $X \in \mathcal{C}(\beta_j)$,

1. $M_j^{\hat{X}} \subseteq D_{i_{\hat{X}}}$, and
2. $\big|M_j^{\hat{X}}\big| = \alpha_X(j)$.

Since $\sum_{j\in I:X\in\mathcal{C}(\beta_j)}\alpha_X(j) \le v_X + \sum_{d\in\cup_{j\in I}D_j:\beta_d=X}\alpha(d)$, there exists a set of donors $M_{i_{\hat{X}}}^X \subseteq \cup_{j\in I\cup\{b\}}D_j$ such that

1. $\beta_d = X$ for every $d \in M_{i_{\hat{X}}}^X$,

[52] Since we do not consider efficiency in this proof, we do not need to assume that $\hat{\succeq}$ is aligned with the patients' preferences.

2. $\alpha(d) = 1$ for every $d \in M^X_{i_{\hat{X}}} \setminus D_b$, and
3. $\left|M^X_{i_{\hat{X}}}\right| = \sum_{j\in I: X\in\mathcal{C}(\beta_j)} \left|M^{\hat{X}}_j\right|$.

Then we construct a matching M for $\hat{D}$ as follows:

- for each $j \in I$, $M_j = \left(\cup_{X\in\mathcal{C}(\beta_j)} M^{\hat{X}}_j\right) \cup \{d \in D_j : \alpha(d) = 0\}$,
- for each $X \in \mathcal{B}$, $M_{i_{\hat{X}}} = M^X_{i_{\hat{X}}} \cup \Big(D_{i_{\hat{X}}} \setminus \big(\cup_{j\in I:X\in\mathcal{C}(\beta_j)} M^{\hat{X}}_j\big)\Big)$, and
- $M_b = \hat{\mathbf{D}} \setminus \big((\cup_{j\in I} M_j) \cup (\cup_{X\in\mathcal{B}} M_{i_{\hat{X}}})\big)$.

Therefore, each patient $j \in I$ is matched with $\alpha_X(j)$ dummy donors of type $\hat{X}$ for every $X \in \mathcal{C}(\beta_j)$ (recall that for the extended problem, $\hat{X} \in \hat{\mathcal{C}}(\beta_j)$), and j's own donor d is matched with j if and only if $\alpha(d) = 0$. Moreover, for each dummy patient $i_{\hat{X}}$, the number of X donors from $I \cup \{b\}$ matched with her is equal to the number of her $\hat{X}$ donors that are not matched with her (recall that $\hat{\mathcal{C}}(\hat{X}) = \{X\}$). Hence, M is a well-defined matching for $\hat{D}$ and it is welfare equivalent to α.

Part 2. On the other hand, let $M \in \mathcal{M}(\hat{D})$. Construct α as follows:

- for each $j \in I$ and $X \in \mathcal{C}(\beta_j)$, let $\alpha_X(j) = \left|\{d \in M_j \setminus D_j : \beta_d \in \{X, \hat{X}\}\}\right|$, and
- for each $j \in I$ and $d \in D_j$, let $\alpha(d) = 0$ if $d \in M_j$, and $\alpha(d) = 1$ if $d \notin M_j$.

If α is an allocation for D, then it is straightforward to show that it is welfare equivalent to M. To show that α is a well-defined allocation, we only need to verify Condition 1 in the definition of an allocation: for any blood type $X \in \mathcal{B}$,

$$\begin{aligned}\sum_{j\in I:X\in\mathcal{C}(\beta_j)} \alpha_X(j) &= \sum_{j\in I:X\in\mathcal{C}(\beta_j)} \left|\{d \in M_j \setminus D_j : \beta_d = X\}\right| + \sum_{j\in I:X\in\mathcal{C}(\beta_j)} \left|M_j \cap D_{i_{\hat{X}}}\right| \\ &\leq \sum_{j\in I:X\in\mathcal{C}(\beta_j)} \left|\{d \in M_j \setminus D_j : \beta_d = X\}\right| + \left|\{d \in M_{i_{\hat{X}}} : \beta_d = X\}\right| \\ &\leq \sum_{j\in I} \left|\{d \in D_j \setminus M_j : \beta_d = X\}\right| + v_X \\ &= \sum_{d\in\cup_{j\in I} D_j:\beta_d=X} \alpha(d) + v_X\end{aligned}$$

where the first inequality follows from the construction of $\mathcal{S}_{i_{\hat{X}}}(D_{i_{\hat{X}}})$, as well as the fact that $\hat{\mathcal{C}}(\hat{X}) = \{X\}$. ∎

Let f be an optimal mechanism induced by $\succeq$. Given an extended schedule profile $\hat{w} = (r_i, s_i)_{i\in\hat{I}\setminus\{b\}}$, let $\hat{w}_I = (r_i, s_i)_{i\in I}$ denote the restriction of $\hat{w}$ to I. It is straightforward to show that there exists a complete, transitive, antisymmetric and responsive preference relation $\hat{\succeq}$ over the extended schedule profiles such that for every $\hat{w}$ and $\hat{w}'$, $\hat{w} \mathrel{\hat{\succ}} \hat{w}'$ if $\hat{w}_I \succ \hat{w}'_I$.[53] Let F be the optimal rule induced by $\hat{\succeq}$. We want to show that for any

[53] For example, we can define $\hat{\succeq}$ as follows. List the dummy patients according to some ordering such that $\{i_{\hat{X}}\}_{X\in\mathcal{B}} = \{i_1, \ldots, i_{|\mathcal{B}|}\}$. For any $\hat{w} = (r_i, s_i)_{i\in\hat{I}\setminus\{b\}}$ and $\hat{w}' = (r'_i, s'_i)_{i\in\hat{I}\setminus\{b\}}$, let $\hat{w} \mathrel{\hat{\succeq}} \hat{w}'$ if

$D \in \mathcal{D}$, $f(D)$ and $F(\hat{D})$ are welfare equivalent.

Let $D \in \mathcal{D}$. By the claim above, there exists $\alpha \in \mathcal{A}(D)$ that is welfare equivalent to $F(\hat{D})$. By the definition of f, $w(f(D)) \succeq w(\alpha)$, where $w(\alpha) = \hat{w}_I(F(\hat{D}))$. On the other hand, there exists $M \in \mathcal{M}(\hat{D})$ that is welfare equivalent to $f(D)$. By the definition of F, $\hat{w}(F(\hat{D})) \mathrel{\hat{\succeq}} \hat{w}(M)$. Then by the property of $\hat{\succeq}$ we have $\hat{w}_I(F(\hat{D})) \succeq \hat{w}_I(M)$, where $\hat{w}_I(M) = w(f(D))$. It has been shown that $w(f(D)) \succeq \hat{w}_I(F(\hat{D}))$ and $\hat{w}_I(F(\hat{D})) \succeq w(f(D))$, which implies that $w(f(D)) = \hat{w}_I(F(\hat{D}))$. Therefore, $f(D)$ and $F(\hat{D})$ are welfare equivalent. ■

The proof of the donor monotonicity of the optimal rules relies on comparing two matchings for two extended problems and constructing two new ones based on the differences between the matches of the patients, respectively. We introduce the following graph theoretical concepts that are central to the proof.

Let $\hat{D}$ and $\hat{D}'$ be two extended problems such that $D'_i \subseteq D_i$ for every $i \in I$. For ease of exposition we also write $D'_{i_{\hat{X}}} = D_{i_{\hat{X}}}$ for every $X \in \mathcal{B}$ and $D'_b = D_b$. Given a matching M for $\hat{D}$ and a matching M' for $\hat{D}'$, a **cycle from M to M'** is a directed graph of patients and donors in which each patient/donor points to the next donor/patient, and is denoted as a list $C = (i_1, d_1, \ldots, i_{\bar{t}}, d_{\bar{t}})$, $\bar{t} \geq 2$, such that for each $t \in \{1, \ldots, \bar{t}\}$ (let $i_{\bar{t}+1} = i_1$ and $d_0 = d_{\bar{t}}$):

1. $i_t \in \hat{I}$, $d_t \in M'_{i_t} \setminus M_{i_t}$ and $d_t \in M_{i_{t+1}}$.
2. If $i_t \neq b$, $d_{t-1} \in D_{i_t}$, and $d_t \notin D_{i_t}$, then
$$(|M_{i_t} \setminus D_{i_t}|+1, |D_{i_t} \setminus M_{i_t}|+1) \in \mathcal{S}_{i_t}(D_{i_t}) \quad \text{and} \quad (|M'_{i_t} \setminus D'_{i_t}|-1, |D'_{i_t} \setminus M'_{i_t}|-1) \in \mathcal{S}_{i_t}(D'_{i_t}).$$
3. If $i_t \neq b$, $d_{t-1} \notin D_{i_t}$, and $d_t \in D_{i_t}$, then
$$(|M_{i_t} \setminus D_{i_t}|-1, |D_{i_t} \setminus M_{i_t}|-1) \in \mathcal{S}_{i_t}(D_{i_t}) \quad \text{and} \quad (|M'_{i_t} \setminus D'_{i_t}|+1, |D'_{i_t} \setminus M'_{i_t}|+1) \in \mathcal{S}_{i_t}(D'_{i_t}).$$
4. If $i_t = i_{t'} = i$ for some $t' \neq t$, then either
   - $d_t, d_{t-1} \in D_i$ and $d_{t'}, d_{t'-1} \notin D_i$, or
   - $d_t, d_{t-1} \notin D_i$ and $d_{t'}, d_{t'-1} \in D_i$.

In a cycle C from M to M', each patient points to a donor that she is matched with under M' but not under M, while each donor points to the patient that she is matched with under M. Note that each donor in the cycle must be in both extended problems, $\hat{D}$ and $\hat{D}'$. Starting from the base matching M, we can assign each patient in the cycle the donor she points to (who is one of her M' matches) instead of the donor she is pointed by (who is one of her M matches). That is, for each $t \in \{1, \ldots, \bar{t}\}$, add d_t to M_{i_t} and

$\hat{w}_I \succ \hat{w}'_I$, or, $\hat{w}_I = \hat{w}'_I$ and there exists some $k \in \{1, \ldots, |\mathcal{B}|\}$ such that (1) $r_{i_k} > r'_{i_k}$, or, $r_{i_k} = r'_{i_k}$ and $s_{i_k} < s'_{i_k}$, and (2) $r_{i_\ell} = r'_{i_\ell}$ and $s_{i_\ell} = s'_{i_\ell}$ for all $\ell < k$. Moreover, for every $\hat{w}$, let $\hat{w} \mathrel{\hat{\succeq}} \hat{w}$.

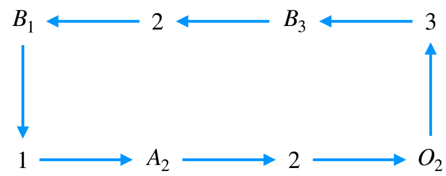

Figure 12: Suppose that $I = \{1, 2, 3\}$, with $\beta_1 = A$, $\beta_2 = B$ and $\beta_3 = O$, $\hat{D} = \hat{D}'$, and the donor sets are given by $D_1 = \{B_1\}$, $D_2 = \{A_2, O_2\}$, $D_3 = \{B_3\}$ and $D_b = \emptyset$, where a type-X donor of a patient i is denoted as X_i. For simplicity, we omit the dummy patients. For every $i \in I$, $\overline{n}_i = 1$, $\underline{n}_i = 0$ and the exchange rate is one-for-one. Assume ABO-identical transfusion. Consider the following two matchings M and M': $M_1 = \{B_1\}$, $M_2 = \{A_2, B_3\}$, $M_3 = \{O_2\}$ and $M_b = \emptyset$; $M'_1 = \{A_2\}$, $M'_2 = \{O_2, B_1\}$, $M'_3 = \{B_3\}$ and $M'_b = \emptyset$. The above graph gives a cycle C from M to M', and we have $M + C = M'$ and $M' - C = M$.

remove d_{t-1} from M_{i_t}. Condition 1 above guarantees that this leads to a well-defined function, which we denote as $M + C$ and satisfies Conditions 1 and 2 in the definition of a matching (for $\hat{D}$). The patients involved in the cycle may not be distinct. But Condition 4 above says that if a patient $i \in \hat{I} \setminus \{b\}$ appears twice in the cycle, then her schedule is not affected by the exchanges, i.e., the amount of blood received and the amount of blood supplied remain the same. Note that this condition also implies that any patient cannot appear more than twice in the cycle. Finally, if a patient $i \in \hat{I} \setminus \{b\}$ is assigned a different schedule under $M + C$ than under M, then she appears only once in the cycle, and she either receives one more unit and supplies one more unit, or receives one less unit and supplies one less unit. Then Conditions 2 and 3 above imply Condition 3 in the definition of a matching. Therefore $M + C$ is a matching for $\hat{D}$. Similarly, we could instead start from M' and assign each patient in the cycle the donor she is pointed by (who is one of her M matches) instead of the donor she points to (who is one of her M' matches). That is, for each $t \in \{1, \ldots, \bar{t}\}$, add d_{t-1} to M'_{i_t} and remove d_t from M'_{i_t}. These exchanges also lead to a well-defined matching for $\hat{D}'$, denoted as $M' - C$. In Figure 12, we give an example of a cycle and the construction of new matchings using this cycle.

It is wise to note that the cycle operations do not necessarily make all patients involved better off or worse off. Instead, they generate new matchings that are closer to each other in terms of the matches of the patients.

Another concept similar to a cycle is a chain. A **chain from** M **to** M' is a directed graph of patients and donors in which each patient/donor points to the next donor/patient in the chain, and is represented as a list $C = (i_1, d_1, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1}, i_{\bar{t}})$, $\bar{t} \geq 2$, such that

1. For every $t \in \{1, \ldots, \bar{t}\}$, $i_t \in \hat{I}$ such that if $i_t = b$ then $t \in \{1, \bar{t}\}$, and $i_1 \neq i_{\bar{t}}$.

2. For every $t \in \{1, \ldots, \bar{t}-1\}$, $d_t \in M'_{i_t} \setminus M_{i_t}$ and $d_t \in M_{i_{t+1}}$.
3. For every $t \in \{2, \ldots, \bar{t}-1\}$, if $d_{t-1} \in D_{i_t}$ and $d_t \notin D_{i_t}$, then
$$(\left|M_{i_t} \setminus D_{i_t}\right|+1, \left|D_{i_t} \setminus M_{i_t}\right|+1) \in \mathcal{S}_{i_t}(D_{i_t}) \quad \text{and} \quad (\left|M'_{i_t} \setminus D'_{i_t}\right|-1, \left|D'_{i_t} \setminus M'_{i_t}\right|-1) \in \mathcal{S}_{i_t}(D'_{i_t}).$$
4. For every $t \in \{2, \ldots, \bar{t}-1\}$, if $d_{t-1} \notin D_{i_t}$, and $d_t \in D_{i_t}$, then
$$(\left|M_{i_t} \setminus D_{i_t}\right|-1, \left|D_{i_t} \setminus M_{i_t}\right|-1) \in \mathcal{S}_{i_t}(D_{i_t}) \quad \text{and} \quad (\left|M'_{i_t} \setminus D'_{i_t}\right|+1, \left|D'_{i_t} \setminus M'_{i_t}\right|+1) \in \mathcal{S}_{i_t}(D'_{i_t}).$$
5. If $i_{\bar{t}} \neq b$, then
$$(\left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|, \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|+1) \in \mathcal{S}_{i_{\bar{t}}}(D_{i_{\bar{t}}}) \quad \text{and} \quad (\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right|, \left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right|-1) \in \mathcal{S}_{i_{\bar{t}}}(D'_{i_{\bar{t}}})$$
when $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$, and
$$(\left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|-1, \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|) \in \mathcal{S}_{i_{\bar{t}}}(D_{i_{\bar{t}}}) \quad \text{and} \quad (\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right|+1, \left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right|) \in \mathcal{S}_{i_{\bar{t}}}(D'_{i_{\bar{t}}})$$
when $d_{\bar{t}-1} \notin D_{i_{\bar{t}}}$.
6. If $i_1 \neq b$, then
$$(\left|M_{i_1} \setminus D_{i_1}\right|, \left|D_{i_1} \setminus M_{i_1}\right|-1) \in \mathcal{S}_{i_1}(D_{i_1}) \quad \text{and} \quad (\left|M'_{i_1} \setminus D'_{i_1}\right|, \left|D'_{i_1} \setminus M'_{i_1}\right|+1) \in \mathcal{S}_{i_1}(D'_{i_1})$$
when $d_1 \in D_{i_1}$, and
$$(\left|M_{i_1} \setminus D_{i_1}\right|+1, \left|D_{i_1} \setminus M_{i_1}\right|) \in \mathcal{S}_{i_1}(D_{i_1}) \quad \text{and} \quad (\left|M'_{i_1} \setminus D'_{i_1}\right|-1, \left|D'_{i_1} \setminus M'_{i_1}\right|) \in \mathcal{S}_{i_1}(D'_{i_1})$$
when $d_1 \notin D_{i_1}$.
7. If $i_t = i_{t'} = i$ for some t, t' such that $1 < t < t' < \bar{t}$, then either
   - $d_t, d_{t-1} \in D_i$ and $d_{t'}, d_{t'-1} \notin D_i$, or,
   - $d_t, d_{t-1} \notin D_i$ and $d_{t'}, d_{t'-1} \in D_i$.

   If $i_{\bar{t}} = i_t = i$ for some t such that $1 < t < \bar{t}$, then either
   - $d_t, d_{t-1} \in D_i$ and $d_{\bar{t}-1} \notin D_i$, or
   - $d_t, d_{t-1} \notin D_i$ and $d_{\bar{t}-1} \in D_i$.

   If $i_1 = i_t = i$ for some t such that $1 < t < \bar{t}$, then either
   - $d_t, d_{t-1} \in D_i$ and $d_1 \notin D_i$, or
   - $d_t, d_{t-1} \notin D_i$ and $d_1 \in D_i$.

A chain differs from a cycle as the last element of a chain is a patient and she does not point to any donor. We refer to this patient, $i_{\bar{t}}$, as the **head** of the chain. As a result there is no donor pointing back to i_1 whom we refer to as the **tail** of the chain. The head and the tail of the chain cannot be the same, and the blood bank b can appear only as the head or the tail (Condition 1).

Similar to the case of a cycle, given a chain C from M to M', we can construct a new matching, denoted as $M + C$, for $\hat{D}$ as follows: starting from M, for each t such that $1 \leq t \leq \bar{t}-1$, remove d_t from $M_{i_{t+1}}$ and add it to M_{i_t}. Condition 7 above implies that any patient cannot appear more than twice in a chain. Moreover, if a patient $i \in \hat{I} \setminus \{b\}$

Figure 13: Suppose that $I = \{1, 2, 3\}$ with $\beta_1 = \beta_2 = A$ and $\beta_3 = B$. The donor sets in two extended problems $\hat{D}$ and $\hat{D}'$ are given by $D_1 = \{B_1\}$, $D'_1 = \emptyset$, $D_2 = D'_2 = \emptyset$, $D_3 = D'_3 = \{A_3\}$ and $D_b = \{A_b, A'_b, B_b\}$, where X_i (or X'_i) denotes a type-X donor of patient i. For simplicity, we omit the dummy patients. For every $i \in I$, $\overline{n}_i = 2$, $\underline{n}_i = 0$ and the feasible schedules are such that the amount supplied does not exceed the amount received. Assume ABO-identical transfusion. Consider the matching M for $\hat{D}$, where $M_1 = \{A'_b\}$, $M_2 = \{A_b\}$, $M_3 = \{A_3, B_1, B_b\}$ and $M_b = \emptyset$, and the matching M' for $\hat{D}'$, where $M'_1 = \{A_b, A'_b\}$, $M'_2 = \{A_3\}$, $M'_3 = \{B_b\}$ and $M'_b = \emptyset$. There does not exist a cycle from M to M', but the above graph gives a chain C from M to M'. Then $M + C$ is a matching for $\hat{D}$, where $(M + C)_1 = \{A_b, A'_b\}$, $(M + C)_2 = \{A_3\}$, $(M + C)_3 = \{B_1, B_b\}$ and $(M + C)_b = \emptyset$. Moreover, $M' - C$ is a matching for $\hat{D}'$, where $(M' - C)_1 = \{A'_b\}$, $(M' - C)_2 = \{A_b\}$, $(M' - C)_3 = \{A_3, B_b\}$ and $(M' - C)_b = \emptyset$.

is assigned a different schedule under $M + C$ than under M, and she appears twice in the chain, then she must appear exactly once as the head or the tail, and only this appearance as the head or the tail affects her schedule. Then Conditions 3, 4, 5, 6 ensure that the schedule of each patient $i \in \hat{I} \setminus \{b\}$ under $M + C$ is indeed feasible. In particular, Conditions 3 and 4 are similar to those of a cycle, while Conditions 5 and 6 deal with special considerations for the head and tail patients. On the other hand, we can also construct a new matching, denoted as $M' - C$, for $\hat{D}'$ as follows: starting from M', for each $1 \leq t \leq \bar{t} - 1$, remove d_t from M'_{i_t} and add it to $M'_{i_{t+1}}$. See Figure 13 for an example of a chain and how new matchings are constructed using this chain.

Unlike in a cycle addition or removal, in the chain operations the number of donors that a patient is matched with only stays the same if she is neither the head nor the tail.[54] Thus, the chain operations change the overall balance of the base matching, while cycle operations do not. The cycle operations would be all we needed if we were dealing with the one-for-one exogenous exchange rate. However, the chain operations play an important role in the general case with endogenously determined exchange rates.

The following observation is straightforward to show from the construction.

Observation 1. *Let C be a cycle or a chain from $M \in \mathcal{M}(\hat{D})$ to $M' \in \mathcal{M}(\hat{D}')$. For every $i \in \hat{I} \setminus \{b\}$, we have*

$$\left|(M + C)_i \setminus D_i\right| - \left|M_i \setminus D_i\right| = \left|M'_i \setminus D'_i\right| - \left|(M' - C)_i \setminus D'_i\right| \in \{-1, 0, 1\},$$

and

$$\left|D_i \setminus (M + C)_i\right| - \left|D_i \setminus M_i\right| = \left|D'_i \setminus M'_i\right| - \left|D'_i \setminus (M' - C)_i\right| \in \{-1, 0, 1\}.$$

[54]The tail gains an additional donor after the chain addition and loses one donor after the chain removal. On the other hand, the head loses one matched donor after the chain addition while she gains an additional matched donor after the chain removal.

Patients:	1 (A)	2 (A)	3 (B)	4 (O)	5 (AB)	6 (A)	7 (O)	b
Donors:	B_1 B_1' AB_1 $\boxed{\boldsymbol{O_1}}$	B_2	A_3	A_4	A_5 O_5	AB_6	A_7	A_b A_b' O_b
$(\underline{\boldsymbol{n}}_i, \overline{\boldsymbol{n}}_i)$ **:**	$(0,3)$	$(1,3)$	$(0,3)$	$(0,3)$	$(0,3)$	$(0,3)$	$(0,3)$	
$\boldsymbol{M}$	B_1 AB_1 A_7 A_b'	A_b	A_3	A_4 O_b	A_5 O_5	AB_6	O_1	B_1' B_2
$\boldsymbol{M'}$	B_1' A_7 A_b' A_b	B_2 A_3 A_4	B_1	O_b	AB_1 AB_6	A_5	O_5	$\emptyset$
$\boldsymbol{M''}$	B_1 O_1 A_7 A_b'	A_b	A_3	A_4 O_b	A_5 AB_1	AB_6	O_5	B_1' B_2

Table 2: The patients, their donors, the minimum guarantees and the maximum needs for Example 2. When Patient 1 truthfully reports his donor set, the matching M is obtained. When he conceals his donor O_1, the matching M' is obtained, in which he receives more blood. M'' is another matching that we explain in the example.

In the remaining of the proof of Theorem 2, we show two lemmata. The first one, Lemma 3, is the most crucial result behind the proof of the theorem. It gives a general necessary condition for any rule that is not donor monotonic. Using this result, we show every optimal rule is donor monotonic (Lemma 4), which concludes the proof.

Lemma 3. *Consider any $D, D' \in \mathcal{D}$ and $i \in I$ such that $D_i' \subseteq D_i$, $\left|D_i \setminus D_i'\right| = 1$, and $D_j' = D_j$ for every $j \in I \setminus \{i\}$. If $M \in \mathcal{M}(\hat{D})$, $M' \in \mathcal{M}(\hat{D}')$, and $\left|M_i' \setminus D_i'\right| > \left|M_i \setminus D_i\right|$, then there exists a cycle or a chain from M to M'.*

The proof of this lemma is rather involved. We illustrate the ideas behind the proof using an example first. The example only demonstrates substantially different *cases* in the construction of a cycle or a chain in the proof, as some of the considered cases use similar constructions.

Example 2. Suppose that $I = \{1, \ldots, 7\}$. We omit the dummy patients for simplicity. The first row in Table 2 gives the blood type of each real patient $i \in I$. The second row gives the donor set D_i for each $i \in I \cup \{b\}$, where X_i (or X_i') denotes a type-X donor of patient i. Let $\overline{n}_i = 3$ for every $i \in I$, $\underline{n}_2 = 1$ and $\underline{n}_i = 0$ for every $i \in I \setminus \{2\}$. Assume ABO-identical transfusion.

We will also consider the situation in which Patient 1 conceals his donor O_1.[55] Let

$$D_1' = D_1 \setminus \{O_1\},$$

and $D_i' = D_i$ for every $i \in I \setminus \{1\}$. Finally, for every $i \in I$ and every $D_i'' \in \mathcal{D}_i$, let

$$\mathcal{S}_i(D_i'') = \{(r, s) : \underline{n}_i \leq r \leq \overline{n}_i, 0 \leq s \leq \left|D_i''\right|, s \leq r\}.$$

The last three rows in Table 2 specify three matchings, M, M' and M'', where M and M'' are matchings for $\hat{D}$ and M' is a matching for $\hat{D}'$. Given that Patient 1 receives

[55] Assume that the patients are male and the donors are female in this example.

more blood under M' than under M, we discuss how to find a cycle or a chain from M to M' using an iterative "pointing procedure from M to M'" that is formally defined in the proof of Lemma 3. At each step of the procedure, a patient points to a donor that he is matched with under M' but not under M, then this donor points to the patient that she is matched with under M.

The procedure starts with the patient who shrank his donor set. Since he receives more blood under M', there is a donor in $M' \setminus M$ that is not his own. He points to one such donor. In this example Patient 1 first points to A_b. Then A_b points to Patient 2, who she is matched with under M. Although B_2 is a donor matched with Patient 2 under M' but not under M, we do not let Patient 2 point to B_2, because

$$(\left|M_2 \setminus D_2\right| - 1, \left|D_2 \setminus M_2\right| - 1) = (0, 0) \notin \mathcal{S}_2(D_2).$$

That is, if we eventually execute this supposed swap of donors from M, the outcome is not a matching as the schedule of Patient 2 is not feasible. Generally, when a patient is pointed by a donor that is not his own (respectively, his own donor), we always first check whether this patient can point to a donor that is not his own (respectively, his own donor), such that the exchanges in the cycle or chain would not affect the patient's schedule. Therefore, we let Patient 2 point to A_3 or A_4. Suppose that Patient 2 points to A_3, then A_3 points to Patient 3. As discussed before, generally, when a patient is pointed by his own donor, we check whether he can point to his own donor. If this is not possible, then he must supply more blood under M' and there are two possible cases:

- If he also receives more blood under M', then we let him point to a donor that is not his own so that, by Assumption $1'$, Condition 2 in the definition of a cycle or Condition 3 in the definition of a chain is satisfied.
- If he does not receive more blood under M', then we stop here and by Assumption $1'$, he can be the head of a chain (i.e., Condition 5 in the definition of a chain is satisfied).

In the example, Patient 3 cannot point to his own donor and he receives more blood under M', so we let him point to B_1. Then B_1 points to Patient 1, and a cycle is found: see the cycle in Figure 14. This construction corresponds to Case 2 in the proof of Lemma 3.

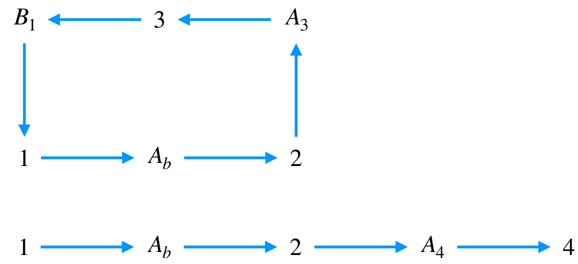

Figure 14: A cycle and a chain from M to M' found using the pointing procedure from M to M' (illustrating Case 2 and Case 3 in the proof of Lemma 3, respectively).

Recall that Patient 2 could also point to A_4. If Patient 2 points to A_4, then A_4 points to Patient 4. Given that Patient 4 cannot point to his own donor and he does not receive more blood under M', we stop here. In this case, a chain is identified as in the graph in Figure 14. This construction corresponds to Case 3 in the proof of Lemma 3. Note that Condition 6 in the definition of a chain is satisfied for Patient 1. This follows from Assumption $2'$ and the fact that his schedules under M and M' are $(2, 2)$ and $(3, 2)$ respectively.

Generally, according to Assumption $2'$, if the manipulating patient receives more blood but does not supply more blood under M', then he can be the tail of a chain. However, if he both receives and supplies more blood under M', then Condition 6 in the definition of a chain may not be satisfied for him. To further discuss this case, we modify the example slightly: Suppose that B_1' is matched with the bank instead of Patient 1 under M', and we change the feasible schedule correspondence of Patient 1 such that

$$\mathcal{S}_1(D_1'') = \{(r, s) : 0 \leq r \leq 3, 0 \leq s \leq \left|D_1''\right|, s = r\}$$

for any D_1''. Then the list $(1, A_b, 2, A_4, 4)$ in Figure 14 is no longer a chain from M to M' since $(\left|M_1 \setminus D_1\right| + 1, \left|D_1 \setminus M_1\right|) = (3, 2) \notin \mathcal{S}_1(D_1)$.

In this case we have to invoke a "backward" pointing procedure. That is, pointing occurs from M' to M: at each step a patient points to a donor that he is matched with under M but not under M', while the donor points to the patient that she is matched with under M'. Then the edge orientation will be reversed to construct a cycle or a chain from M to M'. This corresponds to Case 4 in the proof of Lemma 3.

As Patient 1 supplies less blood under M, there is a donor in $M_1 \setminus M_1'$ that is his own donor, and this donor is not the concealed donor. At the beginning of the pointing procedure from M' to M, Patient 1 points to such a donor. Assume that he points to AB_1. Then AB_1 points to Patient 5. Similar to the previous construction, when a patient is pointed by a donor that is not his own, we first check whether he can point to a donor

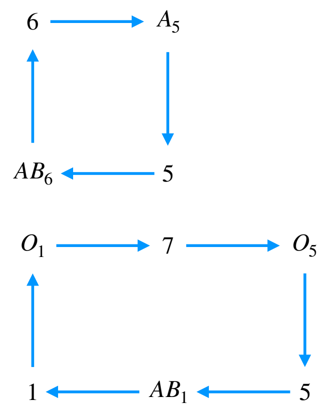

Figure 15: A cycle from M to M' and another directed graph, a *pseudo-cycle* from M to M', in the modified example. Both are constructed by reversing the edge orientation of the graphs found using the pointing procedure from M' to M (illustrating Subcase 4.1 and Subcase 4.5 in the proof of Lemma 3, respectively).

that is not his own. If this is not possible, then he must receive less blood under M and there are two cases:

- If he also supplies less blood under M, let him point to a donor of his own.
- If he does not supply less blood under M, we stop here.

In the example, Patient 5 cannot point to a donor that is not his own and he supplies less blood under M. So Patient 5 points to A_5 or O_5. Suppose that he points to A_5, then A_5 points to Patient 6, Patient 6 points to AB_6, and AB_6 points to Patient 5. After reversing the edge orientation, a cycle from M to M' is found: see the first cycle in Figure 15. This construction corresponds to Subcase 4.1 in the proof of Lemma 3.

On the other hand, if Patient 5 points to O_5, then O_5 points to Patient 7, who points to the concealed donor O_1. Let O_1 point to Patient 1. After reversing the edge orientation, we obtain a list $(1, O_1, 7, O_5, 5, AB_1)$, which is the second graph in Figure 15. However, since $O_1 \notin M_1'$, this is not a cycle from M to M'. We refer to it as a *pseudo-cycle*. Subcase 4.5 in the proof of Lemma 3 deals with this type of situation. We can still carry out the exchanges in the pseudo-cycle based on M, and this leads to the matching M'' for $\hat{D}$. Since Patient 1's schedules under M'' and M are the same and he still receives more blood under M' than under M'', we can repeat the previous analysis and identify a cycle or a chain C from M'' to M', using the pointing procedure from M'' to M' and the pointing procedure from M' to M''. Note that as the donor O_1 is matched with Patient 1 under M'', she will not appear in either pointing procedure. Finally, it can be shown that C is also a cycle or a chain from M to M'. ■

We are ready to prove Lemma 3.

Proof of Lemma 3. Consider two problems $D, D' \in \mathcal{D}$ such that for some patient $i_1 \in I$, $D'_{i_1} \subseteq D_{i_1}$, $\left|D_{i_1} \setminus D'_{i_1}\right| = 1$, and $D'_i = D_i$ for every $i \in I \setminus \{i_1\}$. Suppose that $M \in \mathcal{M}(\hat{D})$, $M' \in \mathcal{M}(\hat{D}')$ and $\left|M'_{i_1} \setminus D'_{i_1}\right| > \left|M_{i_1} \setminus D_{i_1}\right|$. Then there exists a donor $d_1 \notin D_{i_1}$ such that $d_1 \in M'_{i_1} \setminus M_{i_1}$. We will iteratively construct a finite directed graph of patients and donors using the matchings M and M', which is denoted as $(i_1, d_1, i_2, d_2, \ldots)$. It starts with patient i_1, ends with either a patient or a donor, and each node in the list points to the next node.

We refer to this as the *pointing procedure from M to M'*:

<u>Step 1</u>: Let i_1 point to d_1, and d_1 point to $i_2 \in \hat{I}$ such that $d_1 \in M_{i_2}$. If $i_2 = b$ then we stop at i_2 at Step 1, otherwise we continue with Step 2.

<u>Step $t \geq 2$</u>: At the end of Step $t-1$, patient $i_t \in \hat{I} \setminus \{i_1, b\}$ is pointed by d_{t-1} where $d_{t-1} \in M_{i_t} \setminus M'_{i_t}$.

1. If $d_{t-1} \in D_{i_t}$: We have two cases:
   (a) If there exists $d \in D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then at Step t, let i_t point to $d_t = d$, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.[56]
   (b) If there does not exist $d \in D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then $\left|D'_{i_t} \setminus M'_{i_t}\right| > \left|D_{i_t} \setminus M_{i_t}\right|$. We have two subcases:
      i. If $\left|M'_{i_t} \setminus D'_{i_t}\right| > \left|M_{i_t} \setminus D_{i_t}\right|$: Then there exists $d_t \notin D_{i_t}$ such that $d_t \in M'_{i_t} \setminus M_{i_t}$. At Step t, let i_t point to d_t, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.
      ii. If $\left|M'_{i_t} \setminus D'_{i_t}\right| \leq \left|M_{i_t} \setminus D_{i_t}\right|$: Then i_t does not point and stop at i_t at Step $t-1$.
2. If $d_{t-1} \notin D_{i_t}$: We have two cases:
   (a) If there exists $d \notin D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then at Step t, let i_t point to $d_t = d$, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.
   (b) If there does not exist $d \notin D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then $\left|M'_{i_t} \setminus D'_{i_t}\right| < \left|M_{i_t} \setminus D_{i_t}\right|$. We have two subcases:
      i. If $\left|D'_{i_t} \setminus M'_{i_t}\right| < \left|D_{i_t} \setminus M_{i_t}\right|$: Then there exists $d_t \in D_{i_t}$ such that $d_t \in M'_{i_t} \setminus M_{i_t}$. At Step t, let i_t point to d_t, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.

[56] Generally for each $t \geq 1$, such i_{t+1} always exists, since $d_t \in \hat{\mathbf{D}}' \subseteq \hat{\mathbf{D}}$.

ii. If $\left|D'_{i_t} \setminus M'_{i_t}\right| \geq \left|D_{i_t} \setminus M_{i_t}\right|$: Then i_t does not point and stop at i_t at Step $t-1$.

If d_t is constructed, $i_t = i_{\underline{t}} \notin \{i_1, b\}$ for some $\underline{t} < t$, and neither

- $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_t}$ and $d_t, d_{t-1} \notin D_{i_t}$, nor
- $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{i_t}$ and $d_t, d_{t-1} \in D_{i_t}$

holds, then stop at donor d_t at Step t and remove i_{t+1} from the graph construction.

If d_t is constructed, the procedure does not stop at d_t, and $i_{t+1} \in \{i_1, b\}$, then stop at i_{t+1} at Step t.

Otherwise, continue with Step $t+1$.

Note that, according to the above construction, $i_t \neq i_{t+1}$ for any t. Moreover, the procedure stops under four circumstances:

- when some $i \notin \{i_1, b\}$ has appeared before, and the following is not true: she is pointed by and points to her own donors in one instance, and is pointed by and points to donors who are not her own in the other instance,
- when i_1 is pointed,
- when b is pointed,
- when some $i \notin \{i_1, b\}$ does not point.

The first circumstance implies that any patient can be pointed at most three times in the procedure. Hence, the procedure always stops in a finite number of steps.

We consider the following four cases based on these circumstances. Case 1 and Case 2 cover the first two circumstances in order and show the existence of a cycle in each case. Case 3 covers the third and the fourth circumstances together when i_1 does not supply more blood under M' than under M, and shows the existence of a chain. Finally, Case 4 covers the third and the fourth circumstances together when i_1 supplies more blood under M' than under M, and shows the existence of a cycle or a chain. This is the most involved case and we will handle it the last.

Case 1. The procedure stops at $d_{\bar{t}}$ at Step $\bar{t}$.

Then for some $\underline{t} < \bar{t}$, $i_{\underline{t}} = i_{\bar{t}} \notin \{i_1, b\}$ and neither of the following is true:

1. $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\bar{t}}, d_{\bar{t}-1} \notin D_{i_{\underline{t}}}$.
2. $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{i_{\underline{t}}}$ and $d_{\bar{t}}, d_{\bar{t}-1} \in D_{i_{\underline{t}}}$.

We show that $(i_{\underline{t}}, d_{\underline{t}}, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M'.

First, for any t such that $\underline{t} < t \leq \bar{t}-1$, $i_t \notin \{i_1, b\}$, since otherwise the procedure stops at i_t at Step $t-1$. It follows that $D_{i_t} = D'_{i_t}$ for every t such that $\underline{t} \leq t \leq \bar{t}-1$. By the construction of the pointing procedure from M to M', Condition 1 in the definition of a cycle is satisfied. Next, we show Condition 2 and Condition 3.

First, consider any t such that $\underline{t} < t \leq \bar{t}-1$. If $d_{t-1} \in D_{i_t}$ and $d_t \notin D_{i_t}$, then by the construction, we have $\left|M'_{i_t} \setminus D'_{i_t}\right| > \left|M_{i_t} \setminus D_{i_t}\right|$ and $\left|D'_{i_t} \setminus M'_{i_t}\right| > \left|D_{i_t} \setminus M_{i_t}\right|$. Since

$$(\left|M_{i_t} \setminus D_{i_t}\right|, \left|D_{i_t} \setminus M_{i_t}\right|) \in \mathcal{S}_{i_t}(D_{i_t}) \text{ and } (\left|M'_{i_t} \setminus D'_{i_t}\right|, \left|D'_{i_t} \setminus M'_{i_t}\right|) \in \mathcal{S}_{i_t}(D'_{i_t}) = \mathcal{S}_{i_t}(D_{i_t}),$$

it follows from Assumption 1′ that

$$(\left|M_{i_t} \setminus D_{i_t}\right| + 1, \left|D_{i_t} \setminus M_{i_t}\right| + 1) \in \mathcal{S}_{i_t}(D_{i_t}) \text{ and } (\left|M'_{i_t} \setminus D'_{i_t}\right| - 1, \left|D'_{i_t} \setminus M'_{i_t}\right| - 1) \in \mathcal{S}_{i_t}(D'_{i_t}).$$

Similarly, if $d_{t-1} \notin D_{i_t}$ and $d_t \in D_{i_t}$, then by the construction we have $\left|M'_{i_t} \setminus D'_{i_t}\right| < \left|M_{i_t} \setminus D_{i_t}\right|$ and $\left|D'_{i_t} \setminus M'_{i_t}\right| < \left|D_{i_t} \setminus M_{i_t}\right|$. It follows from Assumption 1′ that

$$(\left|M_{i_t} \setminus D_{i_t}\right| - 1, \left|D_{i_t} \setminus M_{i_t}\right| - 1) \in \mathcal{S}_{i_t}(D_{i_t}) \text{ and } (\left|M'_{i_t} \setminus D'_{i_t}\right| + 1, \left|D'_{i_t} \setminus M'_{i_t}\right| + 1) \in \mathcal{S}_{i_t}(D'_{i_t}).$$

Second, consider $i_{\underline{t}}$. Suppose that $d_{\bar{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\underline{t}} \notin D_{i_{\underline{t}}}$. Then either $d_{\underline{t}-1} \in D_{i_{\underline{t}}}$ or $d_{\bar{t}} \notin D_{i_{\underline{t}}}$, as the procedure stops at the donor $d_{\bar{t}}$. Since we have either

- $d_{\bar{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\bar{t}} \notin D_{i_{\underline{t}}}$, or,
- $d_{\underline{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\underline{t}} \notin D_{i_{\underline{t}}}$,

by the construction,

$$\left|M'_{i_{\underline{t}}} \setminus D'_{i_{\underline{t}}}\right| > \left|M_{i_{\underline{t}}} \setminus D_{i_{\underline{t}}}\right| \text{ and } \left|D'_{i_{\underline{t}}} \setminus M'_{i_{\underline{t}}}\right| > \left|D_{i_{\underline{t}}} \setminus M_{i_{\underline{t}}}\right|.$$

Then by Assumption 1′,

$$(\left|M_{i_{\underline{t}}} \setminus D_{i_{\underline{t}}}\right| + 1, \left|D_{i_{\underline{t}}} \setminus M_{i_{\underline{t}}}\right| + 1) \in \mathcal{S}_{i_{\underline{t}}}(D_{i_{\underline{t}}}) \text{ and } (\left|M'_{i_{\underline{t}}} \setminus D'_{i_{\underline{t}}}\right| - 1, \left|D'_{i_{\underline{t}}} \setminus M'_{i_{\underline{t}}}\right| - 1) \in \mathcal{S}_{i_{\underline{t}}}(D'_{i_{\underline{t}}}).$$

That is, Condition 2 in the definition of a cycle is satisfied for $i_{\underline{t}}$. By similar arguments, it can be shown that Condition 3 is also satisfied for $i_{\underline{t}}$.

It remains to show Condition 4. If $i_t = i_{t'}$ and $\underline{t} < t < t' \leq \bar{t}-1$, then either

- $d_t, d_{t-1} \in D_{i_t}$ and $d_{t'}, d_{t'-1} \notin D_{i_t}$, or
- $d_t, d_{t-1} \notin D_{i_t}$ and $d_{t'}, d_{t'-1} \in D_{i_t}$,

since otherwise the procedure stops at $d_{t'}$ at Step t'. Finally, suppose that $i_t = i_{\underline{t}}$ and $\underline{t}+1 < t < \bar{t}-1$. Since the procedure does not stop at d_t at Step t, we have either

(i) $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_t}$ and $d_t, d_{t-1} \notin D_{i_t}$, or,

(ii) $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{i_t}$ and $d_t, d_{t-1} \in D_{i_t}$.

Consider (i) first. Recall that $i_t = i_{\underline{t}} = i_{\bar{t}}$. If $d_{\bar{t}-1} \notin D_{i_t}$, then by the construction of the pointing procedure from M to M', $d_t \notin D_{i_t}$ implies that there exists a donor in $M'_{i_t} \setminus M_{i_t}$ that is not her own, and thus, she should again point to such a donor when she appears for the third time as $i_{\bar{t}}$: $d_{\bar{t}} \notin D_{i_t}$. So we have $d_{\bar{t}}, d_{\bar{t}-1} \notin D_{i_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_t}$, which contradicts to Case 1's assumption. Therefore, $d_{\underline{t}}, d_{\bar{t}-1} \in D_{i_t}$ and $d_t, d_{t-1} \notin D_{i_t}$. Similarly, if (ii) is true, then $d_{\bar{t}-1} \notin D_{i_t}$, since otherwise $d_t \in D_{i_t}$ implies $d_{\bar{t}} \in D_{i_t}$, leading to a contradiction. Hence, $d_{\underline{t}}, d_{\bar{t}-1} \notin D_{i_t}$ and $d_t, d_{t-1} \in D_{i_t}$. This shows that Condition 4 holds, as well.

<u>Case 2.</u> The procedure stops at $i_{\bar{t}}$ at Step $\bar{t}-1$ and $i_{\bar{t}} = i_1$.

To show that $(i_1, d_1, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M', where $d_1 \notin D_{i_1}$, we verify Condition 2 in the definition of a cycle when $d_{\bar{t}-1} \in D_{i_1}$. Since $d_{\bar{t}-1} \in M_{i_1}$ and $d_{\bar{t}-1} \in M'_{i_{\bar{t}-1}}$, $|D_{i_1} \setminus M_{i_1}| < |D_{i_1}|$ and $|D'_{i_1} \setminus M'_{i_1}| > 0$. Then given that $|M'_{i_1} \setminus D'_{i_1}| > |M_{i_1} \setminus D_{i_1}|$, by Assumption 2′, we have

$$(|M_{i_1} \setminus D_{i_1}|+1, |D_{i_1} \setminus M_{i_1}|+1) \in \mathcal{S}_{i_1}(D_{i_1}) \text{ and } (|M'_{i_1} \setminus D'_{i_1}|-1, |D'_{i_1} \setminus M'_{i_1}|-1) \in \mathcal{S}_{i_1}(D'_{i_1}).$$

The other conditions can be shown similarly as in Case 1.

<u>Case 3.</u> The procedure stops at $i_{\bar{t}}$ at Step $\bar{t}-1$, $i_{\bar{t}} \neq i_1$, and $|D'_{i_1} \setminus M'_{i_1}| \leq |D_{i_1} \setminus M_{i_1}|$.

Then either $i_{\bar{t}} = b$ or in the procedure the patient $i_{\bar{t}} \in \hat{I} \setminus \{i_1, b\}$ does not point. We show that $(i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$ is a chain from M to M'. First, $i_t \neq b$ for any $t \in \{2, \ldots, \bar{t}-1\}$ since otherwise the procedure stops at an earlier step. Second, we verify Condition 5 in the definition of a chain. Suppose that $i_{\bar{t}} \neq b$. If $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$, then by the construction, $|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}| > |D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}|$ and $|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}| \leq |M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}|$. Given that $D_{i_{\bar{t}}} = D'_{i_{\bar{t}}}$, by Assumption 1′,

$$(|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}|, |D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}|+1) \in \mathcal{S}_{i_{\bar{t}}}(D_{i_{\bar{t}}}) \text{ and } (|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}|, |D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}|-1) \in \mathcal{S}_{i_{\bar{t}}}(D'_{i_{\bar{t}}}).$$

The case that $d_{\bar{t}-1} \notin D_{i_{\bar{t}}}$ can be shown similarly. Next, Condition 6 follows from the fact that $|M'_{i_1} \setminus D'_{i_1}| > |M_{i_1} \setminus D_{i_1}|$ and $|D'_{i_1} \setminus M'_{i_1}| \leq |D_{i_1} \setminus M_{i_1}|$, as well as Assumption 2′. Finally, we verify Condition 7 for i_1 and $i_{\bar{t}}$. For any $t \in \{2, \ldots, \bar{t}-1\}$, $i_1 \neq i_t$, since otherwise the procedure stops at an earlier step. Suppose that $i_{\bar{t}} = i_t$ for some $t \in \{2, \ldots, \bar{t}-1\}$. Then $i_{\bar{t}} = i_t \neq b$. First consider the case that $d_{\bar{t}-1} \in D_{i_t}$. If $d_t \in D_{i_t}$, then, given that $d_t \in M'_{i_t} \setminus M_{i_t}$, $i_{\bar{t}} = i_t$ should point to this donor (or some other donor of her own) at Step $\bar{t}$, which contradicts to the fact that the pointing procedure stops at $i_{\bar{t}}$. So $d_t \notin D_{i_t}$. Then $d_{t-1} \notin D_{i_t}$, since otherwise $i_{\bar{t}} = i_t$ should point to d_t (or some other donor that is not her own) at Step $\bar{t}$. In the case that $d_{\bar{t}-1} \notin D_{i_t}$, it can be similarly shown that $d_t, d_{t-1} \in D_{i_t}$. These are the crucial conditions to check; the other conditions can be shown similarly as in Case 1.

<u>Case 4.</u> The procedure stops at $i_{\bar{t}}$ at Step $\bar{t}-1$, $i_{\bar{t}} \neq i_1$, and $|D'_{i_1} \setminus M'_{i_1}| > |D_{i_1} \setminus M_{i_1}|$.

In this case, we may not have $(|M_{i_1} \setminus D_{i_1}|+1, |D_{i_1} \setminus M_{i_1}|) \in \mathcal{S}_{i_1}(D_{i_1})$, and hence $(i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$ may not be a chain from M to M'.

Let $j_1 = i_1$. Since $\left|D'_{j_1} \setminus M'_{j_1}\right| > \left|D_{j_1} \setminus M_{j_1}\right|$, there exists a donor $c_1 \in D'_{j_1}$ such that $c_1 \in M_{j_1} \setminus M'_{j_1}$. To find a cycle or a chain, we consider the reverse of the previous construction and use the *pointing procedure from M' to M*. It starts with j_1 pointing to c_1. Then M and D in the pointing procedure from M to M' are replaced with M' and D' respectively, and M' and D' in the pointing procedure from M to M' are replaced with M and D respectively. This pointing procedure from M' to M constructs another directed graph of patients and donors, denoted as $(j_1, c_1, j_2, c_2, \ldots)$, and each node in the list points to the next node in the list. Compared to the previous procedure, there are two slight complications.

First, recall that $D'_{j_1} \subseteq D_{j_1}$ and $\left|D_{j_1} \setminus D'_{j_1}\right| = 1$. We refer to the donor in the set $D_{j_1} \setminus D'_{j_1}$ as the *concealed donor*. If the concealed donor is pointed by j_t at Step $t \geq 2$,[57] let this donor point to $j_{t+1} = j_1$.

Second, there is an additional circumstance in which the procedure stops. At Step $t \geq 2$, if c_t is constructed, $j_t = i_{\underline{t}} \notin \{j_1, b\}$ for some $\underline{t} \in \{2, \ldots, \bar{t} - 1\}$, and neither

- $c_t, c_{t-1} \in D_{j_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{j_t}$, nor
- $c_t, c_{t-1} \notin D_{j_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \in D_{j_t}$

holds, then stop at donor c_t at Step t and remove j_{t+1} from the graph construction.

Then the pointing procedure from M' to M stops under five circumstances, instead of four:

- when some $j \notin \{j_1, b\}$ has appeared before in the pointing procedure from M' to M, and the following is not true: she is pointed by and points to her own donors in one instance, and is pointed by and points to donors who are not her own in the other instance,
- when some $j \notin \{j_1, b\}$ has appeared before in the pointing procedure from M to M', and in this previous appearance she is not $i_{\bar{t}}$. Moreover, the following is not true: she is pointed by and points to her own donors in one instance, and is pointed by and points to donors who are not her own in the other instance,[58]
- when b is pointed,
- when some $j \notin \{j_1, b\}$ does not point,
- when j_1 is pointed.

Due to the first circumstance, the pointing procedure from M' to M also stops in a finite number of steps. Since we are seeking a cycle or a chain from M to M', after the

[57] This can happen in Step t 1.(b)i and Step t 2.(a). Note that the concealed donor does not appear in the pointing procedure from M to M'.

[58] The first and the second circumstances cannot happen at the same time.

procedure stops we reverse the orientation of the constructed edges in $(j_1, c_1, j_2, c_2, \ldots)$.

We consider the following five subcases based on these circumstances. Subcase 4.1 and Subcase 4.2 cover the first two circumstances and show the existence of a cycle in each subcase. Subcase 4.3 covers the third and the fourth circumstances together and shows the existence of a cycle or a chain. Subcase 4.4 covers the fifth circumstance when j_1 is not pointed by the concealed donor, and shows the existence of a cycle. Finally, Subcase 4.5 covers the fifth circumstance when j_1 is pointed by the concealed donor and shows the existence of a cycle or a chain.

Subcase 4.1. The procedure stops at c_t at Step t, for some $\underline{t} < t$, $j_t = j_{\underline{t}} \notin \{j_1, b\}$ and neither of the following is true:

- $c_t, c_{t-1} \in D_{j_t}$ and $c_{\underline{t}}, c_{\underline{t}-1} \notin D_{j_t}$.
- $c_t, c_{t-1} \notin D_{j_t}$ and $c_{\underline{t}}, c_{\underline{t}-1} \in D_{j_t}$.

Then, after reversing the edges in the second directed graph, $(j_t, c_{t-1}, \ldots, j_{\underline{t}+1}, c_{\underline{t}})$ is a cycle from M to M'.

Subcase 4.2. The procedure stops at c_t at Step t, for some $\underline{t} \in \{2, \ldots, \bar{t}-1\}$, $i_{\underline{t}} = j_t \notin \{j_1, b\}$ and neither of the following is true:

- $c_t, c_{t-1} \in D_{j_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{j_t}$.
- $c_t, c_{t-1} \notin D_{j_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \in D_{j_t}$.

We construct a cycle using the first directed graph given by the pointing procedure from M to M', $(i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$, and the second directed graph given by the pointing procedure from M' to M, $(j_1, c_1, \ldots, j_t, c_t)$. Recall that $j_1 = i_1$ and the orientation of the edges in the second graph should be reversed. Then $(j_t, c_{t-1}, \ldots, c_1, i_1, d_1, \ldots, i_{\underline{t}-1}, d_{\underline{t}-1})$ is a cycle from M to M'.

Subcase 4.3. The procedure stops at j_t at Step $t-1$, and $j_t \neq j_1$.

Then either $j_t = b$ or the patient j_t does not point.

If $j_t = i_{\bar{t}} = b$, then $(j_t, c_{t-1}, \ldots, c_1, i_1, d_1, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M'.

If it is not true that $j_t = i_{\bar{t}} = b$, then $(j_t, c_{t-1}, \ldots, c_1, i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$ is a chain from M to M'. To see this, we verify $j_t \neq i_{\bar{t}}$ and Condition 6 in the definition of a chain. First, assume to the contrary, $j_t = i_{\bar{t}}$. Then $j_t = i_{\bar{t}} \in \hat{I} \setminus \{j_1, b\}$. If $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$, then $c_{t-1} \notin D_{i_{\bar{t}}}$, since otherwise in the pointing procedure from M' to M, j_t should point to $d_{\bar{t}-1}$ (or some other donor of her own) at Step t. However, by the construction of the two pointing procedures, $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$ implies $\left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right| > \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|$ and $\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right| \leq \left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|$, while $c_{t-1} \notin D_{i_{\bar{t}}}$ implies $\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right| > \left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|$ and $\left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right| \leq \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|$, contradiction. A similar contradiction can be reached when $d_{\bar{t}-1} \notin D_{i_{\bar{t}}}$. Therefore, $j_t \neq i_{\bar{t}}$. Second, consider Condition 6. If $j_t \neq b$, and $c_{t-1} \in D_{j_t}$, then by the construction we have $\left|D'_{j_t} \setminus M'_{j_t}\right| < \left|D_{j_t} \setminus M_{j_t}\right|$ and $\left|M'_{j_t} \setminus D'_{j_t}\right| \geq \left|M_{j_t} \setminus D_{j_t}\right|$. It follows from Assumption 1′ that

$$(\left|M_{j_t} \setminus D_{j_t}\right|, \left|D_{j_t} \setminus M_{j_t}\right| - 1) \in \mathcal{S}_{j_t}(D_{j_t}) \text{ and } (\left|M'_{j_t} \setminus D'_{j_t}\right|, \left|D'_{j_t} \setminus M'_{j_t}\right| + 1) \in \mathcal{S}_{j_t}(D'_{j_t}).$$

The case that $c_{t-1} \notin D_{j_t}$ can be shown similarly.

Subcase 4.4. The procedure stops at j_t at Step $t-1$, $j_t = j_1$ and $c_{t-1} \notin D_{j_1} \setminus D'_{j_1}$.

Then $(j_t, c_{t-1}, \ldots, j_2, c_1)$ is a cycle from M to M'.

Subcase 4.5. The procedure stops at j_t at Step $t-1$, $j_t = j_1$ and $c_{t-1} \in D_{j_1} \setminus D'_{j_1}$.

Recall that $j_t = j_1 = i_1$ is the patient who concealed her donor, c_{t-1}. First, we have $j_{t'} \in \hat{I} \setminus \{j_t, b\}$ for every $t' \in \{2, \ldots, t-1\}$, since otherwise the procedure stops at an earlier step. As j_t points to the concealed donor $c_{t-1} \notin M'_{j_t}$, $(j_t, c_{t-1}, \ldots, j_2, c_1)$ is not a cycle from M to M'. However, we can still carry out the exchanges in the list $(j_t, c_{t-1}, \ldots, j_2, c_1)$, starting from M: add c_{t-1} to M_{j_t} and remove c_{t-1} from $M_{j_{t-1}}, \ldots,$ add c_1 to M_{j_2} and remove c_1 from M_{j_t}. This leads to a well-defined matching M'' for $\hat{D}$. Since $c_1, c_{t-1} \in D_{j_t}$, $\left|M''_{j_t} \setminus D_{j_t}\right| = \left|M_{j_t} \setminus D_{j_t}\right|$ and $\left|D_{j_t} \setminus M''_{j_t}\right| = \left|D_{j_t} \setminus M_{j_t}\right|$. That is, patient j_t receives and supplies the same amounts of blood under M'' and M.

Given that $\left|M''_{j_t} \setminus D_{j_t}\right| < \left|M'_{j_t} \setminus D'_{j_t}\right|$, we can repeat the previous analysis and identify a cycle or a chain from M'' to M', using the pointing procedure from M'' to M', and possibly the pointing procedure from M' to M''.

Note that the pointing procedure from M'' to M' starts with j_t pointing to some $d \notin D_{j_t}$ with $d \in M'_{j_t} \setminus M''_{j_t}$, and the pointing procedure from M' to M'' starts with j_t pointing to some $c \in D'_{j_t}$ with $c \in M''_{j_t} \setminus M'_{j_t}$. Since $c_{t-1} \notin M'_i$ for any $i \in \hat{I}$, the concealed donor c_{t-1} is not pointed in the pointing procedure from M'' to M'. Moreover, $c_{t-1} \in M''_{j_t}$ implies that c_{t-1} is not pointed in the pointing procedure from M' to M''. Given that c_{t-1} does not appear in either procedure, this recursive Case 4.5 is never reached again, and hence a cycle or a chain C from M'' to M' can be found.

To finish the proof of Lemma 3, it only remains to show that C is also a cycle or a chain from M to M'. We only consider the case that C is a chain, since the proof for the case that C is a cycle is similar and simpler. Let $C = (\ell_1, a_1, \ldots, \ell_{\bar{w}-1}, a_{\bar{w}-1}, \ell_{\bar{w}})$, where $\bar{w} \geq 2$, $a_1, \ldots, a_{\bar{w}-1}$ are donors, and $\ell_1, \ldots, \ell_{\bar{w}}$ are patients. We verify the conditions in the definition of a chain from M to M'.

Since C is a chain from M'' to M', Condition 1 and Condition 7 are trivially satisfied for C to be a chain from M to M'. Consider any $w \in \{1, \ldots, \bar{w}-1\}$. We have $a_w \in M'_{\ell_w} \setminus M''_{\ell_w}$ and $a_w \in M''_{\ell_{w+1}} \setminus M'_{\ell_{w+1}}$. Given that M'' is obtained from M by carrying out the exchanges in the list $(j_t, c_{t-1}, \ldots, j_2, c_1)$, we have $a_w \notin M_{\ell_w}$, since otherwise $a_w \in M_{\ell_w}$ and $a_w \notin M''_{\ell_w}$ imply that ℓ_w is pointed by a_w in the list $(j_t, c_{t-1}, \ldots, j_2, c_1)$ and hence, by the construction of the list, $a_w \notin M'_{\ell_w}$. Similarly, we have $a_w \in M_{\ell_{w+1}}$, since otherwise $a_w \notin M_{\ell_{w+1}}$ and $a_w \in M''_{\ell_{w+1}}$ imply that ℓ_{w+1} points to a_w in the list and hence $a_w \in M'_{\ell_{w+1}}$. Therefore, Condition 2 is satisfied.

To show Conditions 3-6, we need the following result, which follows from the construction of (the reverse of) the list $(j_t, c_{t-1}, \ldots, j_2, c_1)$ in the pointing procedure from M' to M. It essentially says that the schedule of every patient $i \neq b$ under M'' must be "between" her schedules under M and M'.

Observation 2. *For every $i \in \hat{I} \setminus \{b\}$, if $(|M''_i \setminus D_i|, |D_i \setminus M''_i|) \neq (|M_i \setminus D_i|, |D_i \setminus M_i|)$, then $i \neq j_t$, and either*

- $|M'_i \setminus D'_i| > |M_i \setminus D_i|$, $|D'_i \setminus M'_i| > |D_i \setminus M_i|$, *and* $(|M''_i \setminus D_i|, |D_i \setminus M''_i|) = (|M_i \setminus D_i| + 1, |D_i \setminus M_i| + 1)$,
  or
- $|M'_i \setminus D'_i| < |M_i \setminus D_i|$, $|D'_i \setminus M'_i| < |D_i \setminus M_i|$, *and* $(|M''_i \setminus D_i|, |D_i \setminus M''_i|) = (|M_i \setminus D_i| - 1, |D_i \setminus M_i| - 1)$.

Consider any $w \in \{2, \ldots, \bar{w} - 1\}$ such that $a_{w-1} \in D_{\ell_w}$ and $a_w \notin D_{\ell_w}$. Condition 3 is clearly satisfied if $(|M''_{\ell_w} \setminus D_{\ell_w}|, |D_{\ell_w} \setminus M''_{\ell_w}|) = (|M_{\ell_w} \setminus D_{\ell_w}|, |D_{\ell_w} \setminus M_{\ell_w}|)$. Suppose that $(|M''_{\ell_w} \setminus D_{\ell_w}|, |D_{\ell_w} \setminus M''_{\ell_w}|) \neq (|M_{\ell_w} \setminus D_{\ell_w}|, |D_{\ell_w} \setminus M_{\ell_w}|)$. Then $\ell_w \neq j_t$. By the construction of the chain C from M'' to M', we have $|M'_{\ell_w} \setminus D'_{\ell_w}| > |M''_{\ell_w} \setminus D''_{\ell_w}|$ and $|D'_{\ell_w} \setminus M'_{\ell_w}| > |D''_{\ell_w} \setminus M''_{\ell_w}|$. Then by Observation 2, $|M'_{\ell_w} \setminus D'_{\ell_w}| > |M_{\ell_w} \setminus D_{\ell_w}|$ and $|D'_{\ell_w} \setminus M'_{\ell_w}| > |D_{\ell_w} \setminus M_{\ell_w}|$. Hence it follows from Assumption 1′ that Condition 3 is satisfied. Condition 4 can be shown in a similar manner.

Next, consider Condition 5. Suppose that $\ell_{\bar{w}} \neq b$ and $a_{\bar{w}-1} \in D_{\ell_{\bar{w}}}$. For simplicity, denote

- $(|M_{\ell_{\bar{w}}} \setminus D_{\ell_{\bar{w}}}|, |D_{\ell_{\bar{w}}} \setminus M_{\ell_{\bar{w}}}|) = (r, s)$,
- $(|M''_{\ell_{\bar{w}}} \setminus D_{\ell_{\bar{w}}}|, |D_{\ell_{\bar{w}}} \setminus M''_{\ell_{\bar{w}}}|) = (r'', s'')$, and
- $(|M'_{\ell_{\bar{w}}} \setminus D'_{\ell_{\bar{w}}}|, |D'_{\ell_{\bar{w}}} \setminus M'_{\ell_{\bar{w}}}|) = (r', s')$.

Condition 5 is clearly satisfied if $(r, s) = (r'', s'')$. Suppose that $(r, s) \neq (r'', s'')$. Then $\ell_{\bar{w}} \neq j_t$. By the construction of the chain C from M'' to M', we have $s' > s''$ and $r' \leq r''$. Then by Observation 2, $r' > r$, $s' > s$ and $(r'', s'') = (r + 1, s + 1)$. Since $r' > r$ and $r' \leq r'' = r + 1$, we have $r' = r + 1$. By Assumption 1′ and the fact that $r' > r$ and $s' > s$, $(r' - 1, s' - 1) = (r, s' - 1) \in \mathcal{S}_{\ell_{\bar{w}}}(D_{\ell_{\bar{w}}})$. Since $s' - 1 \geq s'' > s$ and $(r, s) \in \mathcal{S}_{\ell_{\bar{w}}}(D_{\ell_{\bar{w}}})$, by Assumption 1′ again, we have $(r, s + 1) \in \mathcal{S}_{\ell_{\bar{w}}}(D_{\ell_{\bar{w}}})$. Finally, $(r', s' - 1) \in \mathcal{S}_{\ell_{\bar{w}}}(D'_{\ell_{\bar{w}}})$ since

C is a chain from M'' to M'. The case that $a_{\bar{w}-1} \notin D_{\ell_{\bar{w}}}$ as well as Condition 6 can be shown similarly. ■

Lemma 4. *Every optimal rule is donor monotonic.*

Proof of Lemma 4. Let F be an optimal rule with respect to $\hat{\succeq}$. To prove that F is donor monotonic, it is sufficient to show that any real patient cannot receive more blood by concealing exactly one donor. Assume to the contrary, there exist $D, D' \in \mathcal{D}$ and $i \in I$ such that $D'_i \subseteq D_i$, $\left|D_i \setminus D'_i\right| = 1$, $D'_j = D_j$ for every $j \in I \setminus \{i\}$, and $\left|F_i(\hat{D}') \setminus D'_i\right| > \left|F_i(\hat{D}) \setminus D_i\right|$. By Lemma 3, there is a cycle or a chain C from $F(\hat{D})$ to $F(\hat{D}')$. We want to first show that $F(\hat{D})$ and $F(\hat{D}) + C$ are welfare equivalent. Suppose that this is not true. Then by the definition of the optimal rule, $\hat{w}(F(\hat{D})) \hat{\succ} \hat{w}(F(\hat{D}) + C)$. We take the component-wise minimum of the two vectors $\hat{w}(F(\hat{D}))$ and $\hat{w}(F(\hat{D}) + C)$: define an extended schedule profile $\hat{w}'$ such that for each component $k \in \{1, \dots, 2(|\hat{I}| - 1)\}$, $\hat{w}'_k = \min\left\{\hat{w}_k(F(\hat{D})), \hat{w}_k(F(\hat{D}) + C)\right\}$. Then

$$\hat{w}' + \big(\hat{w}(F(\hat{D})) - \hat{w}'\big) \hat{\succ} \hat{w}' + \big(\hat{w}(F(\hat{D}) + C) - \hat{w}'\big).$$

By Observation 1, each component of $\hat{w}(F(\hat{D})) - \hat{w}'$ and $\hat{w}(F(\hat{D}) + C) - \hat{w}'$ is either 0 or 1. Hence, by responsiveness,

$$\hat{w}(F(\hat{D})) - \hat{w}' \hat{\succ} \hat{w}(F(\hat{D}) + C) - \hat{w}',$$

and

$$\hat{w}'' + \big(\hat{w}(F(\hat{D})) - \hat{w}'\big) \hat{\succ} \hat{w}'' + \big(\hat{w}(F(\hat{D}) + C) - \hat{w}'\big),$$

where $\hat{w}''$ is defined such that for each $k \in \{1, \dots, 2(|\hat{I}| - 1)\}$, $\hat{w}''_k = \min\big\{\hat{w}_k(F(\hat{D}') - C), \hat{w}_k(F(\hat{D}'))\big\}$. By Observation 1,

$$\hat{w}(F(\hat{D})) - \hat{w}' = \hat{w}(F(\hat{D}') - C) - \hat{w}'', \text{ and } \hat{w}(F(\hat{D}) + C) - \hat{w}' = \hat{w}(F(\hat{D}')) - \hat{w}''.$$

Therefore,

$$\hat{w}(F(\hat{D}') - C) \hat{\succ} \hat{w}(F(\hat{D}')),$$

contradicting to the definition of F. Hence, $F(\hat{D})$ and $F(\hat{D}) + C$ are welfare equivalent. Then by Lemma 3 again, there is a cycle or a chain C' from $F(\hat{D}) + C$ to $F(\hat{D}')$. By similar arguments as before, it can be shown that $(F(\hat{D}) + C) + C'$ and $F(\hat{D}) + C$ are welfare equivalent. Then $(F(\hat{D}) + C) + C'$ and $F(\hat{D})$ are welfare equivalent. This process can be continued infinitely, which leads to a contradiction since each additional cycle or chain addition generates a matching that is closer to $F(\hat{D}')$. ■

A.3 Proof of Theorem 4

We first show that, given any optimal mechanism, if a patient's feasible schedule set becomes weakly more favorable, then she cannot receive less blood. The proof of this part uses the same techniques as those in the proof of Theorem 2. We explain how to modify the previous arguments to prove it. First, we present the following condition regarding different feasible schedule correspondences, which is a counterpart of Assumption 2′.

Assumption 2″. Consider any two profiles of feasible schedule correspondences, $\mathcal{S}$ and $\mathcal{S}'$. For every $i \in I$ and $D_i \in \mathcal{D}_i$, if $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ at D_i, then for any $(r,s) \in \mathcal{S}_i(D_i)$ and any $(r',s') \in \mathcal{S}'_i(D_i)$, we have

1. If $r' > r, s' > 0$ and $s < \left|D_i\right|$, then
$$(r+1, s+1) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r'-1, s'-1) \in \mathcal{S}'_i(D_i).$$
2. If $r' > r$ and $s' \leq s$, then
$$(r+1, s) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r'-1, s') \in \mathcal{S}'_i(D_i).$$

Using arguments similar to those in the proof of Lemma 1, it can be shown that when Assumptions 1 and 2 are satisfied for all feasible schedule correspondences, Assumption 2″ is satisfied.

Second, we use the same construction of extended problems as before. For a given profile of feasible schedule correspondences $\mathcal{S} = (\mathcal{S}_i)_{i \in I}$ and an optimal rule F, let $F\big(\hat{D} \,|\, \mathcal{S}\big)$ denote the outcome matching of F for $\hat{D}$ under $\mathcal{S}$.[59] Since Lemma 2 holds for arbitrary feasible schedule correspondences, we know that for every optimal mechanism f, there exists an optimal rule F such that for any profile of feasible schedule correspondences $\mathcal{S}$ and any $D \in \mathcal{D}$, $f(D \,|\, \mathcal{S})$ and $F(\hat{D} \,|\, \mathcal{S})$ are welfare equivalent. Therefore, for the first part of the proof, it is sufficient to show the following result.

Lemma 5. *Consider any optimal rule F, any $D \in \mathcal{D}$, and any patient $i \in I$. If $\mathcal{S}$ and $\mathcal{S}'$ are two profiles of feasible schedule correspondences such that $\mathcal{S}_j(D_j) = \mathcal{S}'_j(D_j)$ for all $j \in I \setminus \{i\}$, and $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ at D_i, then*
$$\left|F_i\big(\hat{D} \,|\, \mathcal{S}\big) \setminus D_i\right| \geq \left|F_i\big(\hat{D} \,|\, \mathcal{S}'\big) \setminus D_i\right|.$$

To prove this lemma, we need the cycle and chain operations as before. Recall that cycles and chains are defined with respect to two different matchings corresponding to two different donor profiles. We modify their definitions slightly such that they are defined with respect to two different matchings corresponding to the same donor profile but two different profiles of feasible schedule correspondences. Below we give the modified definition of a cycle.

Given a matching M for $\hat{D}$ under $\mathcal{S}$, and a matching M' for $\hat{D}$ under $\mathcal{S}'$, a cycle from M to M' is a directed graph of patients and donors in which each patient/donor points

[59] Every dummy patient $i_{\hat{X}}$ has the fixed feasible schedule correspondence that is induced by the one-for-one exchange rate.

to the next donor/patient, and is denoted as a list $C = (i_1, d_1, \ldots, i_{\bar{t}}, d_{\bar{t}})$, $\bar{t} \geq 2$, such that for each $t \in \{1, \ldots, \bar{t}\}$ (let $i_{\bar{t}+1} = i_1$ and $d_0 = d_{\bar{t}}$):

1. $i_t \in \hat{I}$, $d_t \in M'_{i_t} \setminus M_{i_t}$ and $d_t \in M_{i_{t+1}}$.
2. If $i_t \neq b$, $d_{t-1} \in D_{i_t}$, and $d_t \notin D_{i_t}$, then
   $(|M_{i_t} \backslash D_{i_t}|+1, |D_{i_t} \backslash M_{i_t}|+1) \in \mathcal{S}_{i_t}(D_{i_t})$ and $(|M'_{i_t} \backslash D_{i_t}|-1, |D_{i_t} \backslash M'_{i_t}|-1) \in \mathcal{S}'_{i_t}(D_{i_t})$.[60]
3. If $i_t \neq b$, $d_{t-1} \notin D_{i_t}$, and $d_t \in D_{i_t}$, then
   $(|M_{i_t} \backslash D_{i_t}|-1, |D_{i_t} \backslash M_{i_t}|-1) \in \mathcal{S}_{i_t}(D_{i_t})$ and $(|M'_{i_t} \backslash D_{i_t}|+1, |D_{i_t} \backslash M'_{i_t}|+1) \in \mathcal{S}'_{i_t}(D_{i_t})$.
4. If $i_t = i_{t'} = i$ for some $t' \neq t$, then either
   - $d_t, d_{t-1} \in D_i$ and $d_{t'}, d_{t'-1} \notin D_i$, or
   - $d_t, d_{t-1} \notin D_i$ and $d_{t'}, d_{t'-1} \in D_i$.

The definition of a chain is modified in the same way.[61] Then the following result is a counterpart of Lemma 3.

Lemma 6. *Consider any $D \in \mathcal{D}$ and any patient $i \in I$. Suppose that $\mathcal{S}$ and $\mathcal{S}'$ are two profiles of feasible schedule correspondences such that $\mathcal{S}_j(D_j) = \mathcal{S}'_j(D_j)$ for all $j \in I \setminus \{i\}$, and $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ at D_i. If M is a matching for $\hat{D}$ under $\mathcal{S}$, M' is a matching for $\hat{D}$ under $\mathcal{S}'$, and $|M'_i \setminus D_i| > |M_i \setminus D_i|$, then there is a cycle or a chain from M to M'.*

Using Assumptions $1'$ and $2''$, Lemma 6 can be proved in the same way as Lemma 3. Since there is no concealed donor, Case 4.5 in the proof of Lemma 3 cannot happen.

By arguments similar to those in the proof of Lemma 4, Lemma 5 can be proved using Lemma 6. Specifically, we prove by contradiction. Assume that there exist some optimal rule F, $D \in \mathcal{D}$, $i \in I$, $\mathcal{S}$ and $\mathcal{S}'$, such that $\mathcal{S}_j(D_j) = \mathcal{S}'_j(D_j)$ for all $j \in I \setminus \{i\}$, $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ at D_i, and

$$\left|F_i(\hat{D} \,|\, \mathcal{S}) \setminus D_i\right| < \left|F_i(\hat{D} \,|\, \mathcal{S}') \setminus D_i\right|.$$

Then by Lemma 6, there is a cycle or a chain C from $F(\hat{D} \,|\, \mathcal{S})$ to $F(\hat{D} \,|\, \mathcal{S}')$. It can be shown that $F(\hat{D} \,|\, \mathcal{S}) + C$ is welfare equivalent to $F(\hat{D} \,|\, \mathcal{S})$. By Lemma 6 again, there is a cycle or a chain C' from $F(\hat{D} \,|\, \mathcal{S}) + C$ to $F(\hat{D} \,|\, \mathcal{S}')$. Then $(F(\hat{D} \,|\, \mathcal{S}) + C) + C'$ is welfare equivalent to $F(\hat{D} \,|\, \mathcal{S})$. We can continue this process and after a finite number of cycle or chain additions, we have $F(\hat{D} \,|\, \mathcal{S}')$ is welfare equivalent to $F(\hat{D} \,|\, \mathcal{S})$, contradiction.

[60] If i_t is a dummy patient, then $\mathcal{S}_{i_t} = \mathcal{S}'_{i_t}$ is her fixed feasible schedule correspondence that is induced by the one-for-one exchange rate.

[61] As in the case of a cycle, the only changes we make are replacing $\hat{D}'$ with $\hat{D}$, and replacing $\mathcal{S}_i(D'_i)$ with $\mathcal{S}'_i(D_i)$ everywhere these two appear in the definition.

To finish the proof of Theorem 4, it remains to show that, under any optimal mechanism, if a patient's feasible schedule set becomes weakly more favorable, and the amount of blood she receives does not change, then she cannot supply more blood. Assume to the contrary, there exist some optimal mechanism f induced by $\succeq$, $D \in \mathcal{D}$, $i \in I$, $\mathcal{S}$ and $\mathcal{S}'$, such that $\mathcal{S}_j(D_j) = \mathcal{S}'_j(D_j)$ for all $j \in I \setminus \{i\}$, $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ at D_i,

$$f\big(D \,|\, \mathcal{S}\big)(i) = f\big(D \,|\, \mathcal{S}'\big)(i), \text{ and } \sum_{d \in D_i} f\big(D \,|\, \mathcal{S}\big)(d) > \sum_{d \in D_i} f\big(D \,|\, \mathcal{S}'\big)(d).$$

Let $f\big(D \,|\, \mathcal{S}\big)(i) = r$, $\sum_{d \in D_i} f\big(D \,|\, \mathcal{S}\big)(d) = s$ and $\sum_{d \in D_i} f\big(D \,|\, \mathcal{S}'\big)(d) = s'$. Since $s > 0$, $\mathcal{S}_i(D_i) \neq \{(0,0)\}$ and $r \geq \underline{n}_i$. Given that $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ at D_i and $(r, s') \in \mathcal{S}'_i(D_i)$, there exists $s'' \leq s' < s$ such that $(r, s'') \in \mathcal{S}_i(D_i)$. Then, as $(r, s) \in \mathcal{S}_i(D_i)$, by Assumption 1′ we have $(r, s') \in \mathcal{S}_i(D_i)$. Similarly, it can be shown that $(r, s) \in \mathcal{S}'_i(D_i)$. Finally, $(r, s') \in \mathcal{S}_i(D_i)$ implies that $f\big(D \,|\, \mathcal{S}'\big)$ is an allocation for D under $\mathcal{S}$, and hence $w\big(f\big(D \,|\, \mathcal{S}\big)\big) \succ w\big(f\big(D \,|\, \mathcal{S}'\big)\big)$. On the other hand, $(r, s) \in \mathcal{S}'_i(D_i)$ implies that $f\big(D \,|\, \mathcal{S}\big)$ is an allocation for D under $\mathcal{S}'$, and hence $w\big(f\big(D \,|\, \mathcal{S}'\big)\big) \succ w\big(f\big(D \,|\, \mathcal{S}\big)\big)$. Therefore, a contradiction is reached.

Supplemental Material

B The General Multi-unit Exchange Model under Private Information

The main theoretical results in the paper are independent of the blood allocation and transfusion practices, and our model can be used to study the general multi-unit exchange of indivisible objects with compatibility-based preferences over the objects, where for each agent both such preferences and her endowments are private information. To this end, we first reinterpret several elements in the model.

We consider I as a set of **agents**, and $\beta_i \in \mathcal{B}$ as the **type** of agent $i \in I$. For every $i \in I$, each $D_i \in \mathcal{D}_i$ is a set of **objects** initially owned by agent i, i.e., the **endowments** of i, and $\beta_d \in \mathcal{B}$ is the **type** of each object $d \in D_i$. For every $X \in \mathcal{B}$, there are v_X existing objects of type X that are not the endowments of any agent. We assume that for any $i, j \in I$ such that $i \neq j$, $\beta_i \neq \beta_j$, i.e., any two different agents have different types. Every agent $i \in I$ has compatibility-based preferences over the objects and such preferences are represented by the set of types compatible with her type, $\mathcal{C}(\beta_i) \subseteq \mathcal{B}$. Let $\mathcal{C} = \big(\mathcal{C}(\beta_i)\big)_{i \in I}$.[62] In an allocation α for D and $\mathcal{C}$, an agent i receives $\alpha(i)$ compatible objects, and supplies $\sum_{d \in D_i} \alpha(d)$ objects from her endowments.[63] A mechanism f assigns an allocation, denoted as $f\big(D \,|\, \mathcal{C}\big)$, to each combination of endowment profile $D \in \mathcal{D}$ and preference profile $\mathcal{C}$.

As in the case of blood allocation, over-reporting the endowment set is usually infeasible in practice, and hence we only consider the possibility of under-reporting endowments. On the other hand, an agent can report an arbitrary set of compatible types. Based on these two incentive issues, we define the following more general incentive axioms.A mechanism f is **weakly strategy-proof** if for any $i \in I$, $D, D' \in \mathcal{D}$, $\mathcal{C}$ and $\mathcal{C}'$ such that $D'_i \subseteq D_i$, $D_j = D'_j$, $\mathcal{C}(\beta_j) = \mathcal{C}'(\beta_j)$ for all $j \in I \setminus \{i\}$, and $f\big(D' \,|\, \mathcal{C}'\big)_X(i) = 0$ for every $X \in \mathcal{C}'(\beta_i) \setminus \mathcal{C}(\beta_i)$,[64] we have

$$f\big(D \,|\, \mathcal{C}\big)(i) \geq f\big(D' \,|\, \mathcal{C}'\big)(i).$$

[62] Depending on the application, the minimum guarantees can still be utilized as a policy variable. As long as each agent always reports at least one compatible type, we can assume v is large enough such that every agent i can receive at least $\underline{n}_i$ compatible objects, regardless of the reported preferences $\mathcal{C}$ and the reported endowments D.

[63] Note that her preferences over schedules are still assumed to be lexicographic.

[64] We assume that if an agent receives at least one incompatible object, then this outcome is strictly worse than any outcome in which she receives only compatible objects.

A mechanism f is **strategy-proof** if for any $i \in I$, $D, D' \in \mathcal{D}$, $\mathcal{C}$ and $\mathcal{C}'$ such that $D'_i \subseteq D_i$, $D_j = D'_j$, $\mathcal{C}(\beta_j) = \mathcal{C}'(\beta_j)$ for all $j \in I \setminus \{i\}$, and $f\big(D' \,|\, \mathcal{C}'\big)_X(i) = 0$ for every $X \in \mathcal{C}'(\beta_i) \setminus \mathcal{C}(\beta_i)$, we have

$$w_i\big(f\big(D \,|\, \mathcal{C}\big)\big) \; \mathbf{R}_i \; w_i\big(f\big(D' \,|\, \mathcal{C}'\big)\big).$$

Recall that, to incentivize an agent to report her full set of endowments, we require her feasible schedule set to become more favorable as she reports a larger set of endowments (Assumptions 3 and 4). Given that an agent may over-report or under-report her set of compatible types, we do not allow her feasible schedule set to vary with her preferences. That is, for each agent i, once D_i is given, $\mathcal{S}_i(D_i)$ is fixed and does not depend on $\mathcal{C}(\beta_i)$. Under the same assumptions on the feasible schedule correspondences as in Theorem 2, given an optimal mechanism, if an agent under-reports her endowment set and/or misreports her preferences, then she either receives an incompatible object, or receives weakly less compatible objects.

Theorem S.5. *Under Assumptions 1, 2 and 3, every optimal mechanism is weakly strategy-proof.*

Under these assumptions, the exchange rates in this general model can be endogenously determined by the optimal mechanism. However, an agent may be able to under-report her set of compatible types (or endowment set) such that she still receives the same amount of compatible objects, but supplies less endowments. To ensure strategy-proofness, we further need the exchange rates to be exogenous, i.e., for every $i \in I$, $D_i \in \mathcal{D}_i$ and $(r, s) \in \mathcal{S}_i(D_i)$, there does not exist $s' \neq s$ such that $(r, s') \in \mathcal{S}_i(D_i)$. Note that in this case, as explained in Remark 2, the feasible schedule correspondences are two-part tariffs.

Corollary 1. *Under Assumptions 1, 2 and 3, if the exchange rates are exogenous, then every optimal mechanism is strategy-proof.*

B.1 Proof of Theorem S.5

Suppose that Assumptions 1, 2 and 3 are satisfied. Consider an optimal mechanism f, induced by an aggregate preference relation $\succeq$. When an agent both under-reports her endowments and misreports her preferences, we know by Theorem 2 that she receives weakly less compatible objects if she only under-reports her endowments first. Therefore, to prove that f is weakly strategy-proof, we only need to show that if any agent misreports her preferences, then she either receives an incompatible object, or receives weakly less compatible objects. This can be shown in the following two parts, because for an agent i

and her two sets of compatible types $\mathcal{C}(\beta_i)$ and $\mathcal{C}'(\beta_i)$, we have $\mathcal{C}'(\beta_i) = \big(\mathcal{C}(\beta_i) \setminus \mathcal{B}_1\big) \cup \mathcal{B}_2$, where $\mathcal{B}_1 = \mathcal{C}(\beta_i) \setminus \mathcal{C}'(\beta_i)$ and $\mathcal{B}_2 = \mathcal{C}'(\beta_i) \setminus \mathcal{C}(\beta_i)$.

1. If any agent over-reports her set of compatible types, then she either receives an incompatible object, or receives weakly less compatible objects.
2. If any agent under-reports her set of compatible types, then she receives weakly less compatible objects.

We prove the first part by contradiction. Suppose that there exist $i \in I$, $D \in \mathcal{D}$, $\mathcal{C}$ and $\mathcal{C}'$ such that $\mathcal{C}(\beta_i) \subseteq \mathcal{C}'(\beta_i)$, $\mathcal{C}(\beta_j) = \mathcal{C}'(\beta_j)$ for all $j \in I \setminus \{i\}$, $f\big(D \,|\, \mathcal{C}'\big)_X(i) = 0$ for all $X \in \mathcal{C}'(\beta_i) \setminus \mathcal{C}(\beta_i)$, and

$$f\big(D \,|\, \mathcal{C}'\big)(i) > f\big(D \,|\, \mathcal{C}\big)(i).$$

Define an allocation α for D and $\mathcal{C}'$ such that $\alpha(d) = f\big(D \,|\, \mathcal{C}\big)(d)$ for all $d \in \cup_{j \in I} D_j$, $\alpha_X(j) = f\big(D \,|\, \mathcal{C}\big)_X(j)$ for all $j \in I$ and $X \in \mathcal{C}(\beta_j)$, and $\alpha_X(i) = 0$ for all $X \in \mathcal{C}'(\beta_i) \setminus \mathcal{C}(\beta_i)$. Then $w(\alpha) = w\big(f\big(D \,|\, \mathcal{C}\big)\big) \neq w\big(f\big(D \,|\, \mathcal{C}'\big)\big)$. By the definition of the optimal mechanism, we have $w\big(f\big(D \,|\, \mathcal{C}'\big)\big) \succ w\big(f\big(D \,|\, \mathcal{C}\big)\big)$. On the other hand, define an allocation α' for D and $\mathcal{C}$ such that $\alpha'(d) = f\big(D \,|\, \mathcal{C}'\big)(d)$ for all $d \in \cup_{j \in I} D_j$, $\alpha'_X(j) = f\big(D \,|\, \mathcal{C}'\big)_X(j)$ for all $j \in I$ and $X \in \mathcal{C}(\beta_j)$. Then $w(\alpha') = w\big(f\big(D \,|\, \mathcal{C}'\big)\big)$. For D and $\mathcal{C}$, the mechanism designer chooses $f\big(D \,|\, \mathcal{C}\big)$ over α', i.e., $w\big(f\big(D \,|\, \mathcal{C}\big)\big) \succ w\big(f\big(D \,|\, \mathcal{C}'\big)\big)$, contradiction.

To prove the second part, we use the same techniques as those in the proof of Theorem 2. First, for every $D \in \mathcal{D}$ and $\mathcal{C}$, the construction of the extended problem remains the same. Recall that the compatibility for the extended problem $\hat{D}$ is denoted as $\hat{\mathcal{C}}$, which takes into account the dummy types. Let $F(\hat{D} \,|\, \hat{\mathcal{C}})$ denote the outcome matching of an optimal rule F. Then in light of Lemma 2, it is sufficient to show the result for the optimal rules:

Lemma S.7. *Consider any optimal rule F, any $i \in I$, and any $D \in \mathcal{D}$. If $\mathcal{C}$ and $\mathcal{C}'$ are two preference profiles such that $\mathcal{C}'(\beta_i) \subseteq \mathcal{C}(\beta_i)$ and $\mathcal{C}'(\beta_j) = \mathcal{C}(\beta_j)$ for all $j \in I \setminus \{i\}$, then*

$$\Big|F_i\big(\hat{D} \,|\, \hat{\mathcal{C}}\big) \setminus D_i\Big| \geq \Big|F_i\big(\hat{D} \,|\, \hat{\mathcal{C}}'\big) \setminus D_i\Big|.$$

To prove Lemma S.7, we rely on the cycle and chain operations as before. In this context, the definitions of a cycle and a chain have to be modified such that they are defined with respect to two different matchings corresponding to two different preference profiles but the same endowment profile.

Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are two preference profiles such that $\mathcal{C}'(\beta_i) \subseteq \mathcal{C}(\beta_i)$ for all $i \in I$. Let $D \in \mathcal{D}$. Given a matching M for $\hat{D}$ and $\hat{\mathcal{C}}$, and a matching M' for $\hat{D}$ and $\hat{\mathcal{C}}'$, a cycle from M to M' is a directed graph of agents and objects in which each agent/object points to the next object/agent, and is denoted as a list $C = (i_1, d_1, \ldots, i_{\bar{t}}, d_{\bar{t}})$, $\bar{t} \geq 2$, such that for each $t \in \{1, \ldots, \bar{t}\}$ (let $i_{\bar{t}+1} = i_1$ and $d_0 = d_{\bar{t}}$):

1. $i_t \in \hat{I}$, $d_t \in M'_{i_t} \setminus M_{i_t}$ and $d_t \in M_{i_{t+1}}$.
2. If $i_t \neq b$, $d_{t-1} \in D_{i_t}$, and $d_t \notin D_{i_t}$, then
$(|M_{i_t} \setminus D_{i_t}|+1, |D_{i_t} \setminus M_{i_t}|+1) \in \mathcal{S}_{i_t}(D_{i_t})$ and $(|M'_{i_t} \setminus D_{i_t}|-1, |D_{i_t} \setminus M'_{i_t}|-1) \in \mathcal{S}_{i_t}(D_{i_t})$.
3. If $i_t \neq b$, $d_{t-1} \notin D_{i_t}$, and $d_t \in D_{i_t}$, then
$(|M_{i_t} \setminus D_{i_t}|-1, |D_{i_t} \setminus M_{i_t}|-1) \in \mathcal{S}_{i_t}(D_{i_t})$ and $(|M'_{i_t} \setminus D_{i_t}|+1, |D_{i_t} \setminus M'_{i_t}|+1) \in \mathcal{S}_{i_t}(D_{i_t})$.
4. If $i_t = i_{t'} = i$ for some $t' \neq t$, then either
   - $d_t, d_{t-1} \in D_i$ and $d_{t'}, d_{t'-1} \notin D_i$, or
   - $d_t, d_{t-1} \notin D_i$ and $d_{t'}, d_{t'-1} \in D_i$.
5. If $i_t \neq b$ and $d_{t-1} \notin D_{i_t}$, then $\beta_{d_{t-1}} \in \hat{\mathcal{C}}'(\beta_{i_t})$.

A key difference from the original definition of a cycle is that we need an additional condition, Condition 5, which ensures that in the cycle removal operation no agent is assigned an incompatible object, so that $M' - C$ is a well-defined matching for $\hat{D}$ and $\hat{\mathcal{C}}'$. The definition of a chain is modified in the same way.

Lemma S.8. *Consider any $i \in I$ and any $D \in \mathcal{D}$. Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are two preference profiles such that $\mathcal{C}'(\beta_i) \subseteq \mathcal{C}(\beta_i)$ and $\mathcal{C}'(\beta_j) = \mathcal{C}(\beta_j)$ for all $j \in I \setminus \{i\}$. If M is a matching for $\hat{D}$ and $\hat{\mathcal{C}}$, M' is a matching for $\hat{D}$ and $\hat{\mathcal{C}}'$, and $|M'_i \setminus D_i| > |M_i \setminus D_i|$, then there is a cycle or a chain from M to M'.*

Proof of Lemma S.8. Consider any $D \in \mathcal{D}$. Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are two preference profiles such that for some $i_1 \in I$, $\mathcal{C}'(\beta_{i_1}) \subseteq \mathcal{C}(\beta_{i_1})$ and $\mathcal{C}'(\beta_j) = \mathcal{C}(\beta_j)$ for all $j \in I \setminus \{i_1\}$. Moreover, M is a matching for $\hat{D}$ and $\hat{\mathcal{C}}$, M' is a matching for $\hat{D}$ and $\hat{\mathcal{C}}'$, and $|M'_{i_1} \setminus D_{i_1}| > |M_{i_1} \setminus D_{i_1}|$. Then there exists $d_1 \notin D_{i_1}$ such that $d_1 \in M'_{i_1} \setminus M_{i_1}$. As in the proof of Lemma 3, we construct a directed graph $(i_1, d_1, i_2, d_2, \ldots)$ using the pointing procedure from M to M'.[65] We consider the following four cases. The analysis of Cases 1 and 2 is similar as before. The pointing procedure from M' to M is needed in Case 3, but there will not be any "pseudo-cycle". However, a pseudo-cycle may appear in the pointing procedure from M to M' in Case 4.

<u>Case 1.</u> The procedure stops at $d_{\bar{t}}$ at Step $\bar{t}$.

Then a cycle from M to M' can be found.

<u>Case 2.</u> The procedure stops at $i_{\bar{t}}$ at Step $\bar{t}-1$, $i_{\bar{t}} \neq i_1$, and $|D_{i_1} \setminus M'_{i_1}| \leq |D_{i_1} \setminus M_{i_1}|$.

Then $(i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$ is a chain from M to M'.

[65] D' in the original definition of the pointing procedure has to be replaced with D.

Case 3. The procedure stops at $i_{\bar{t}}$ at Step $\bar{t}-1$, $i_{\bar{t}} \neq i_1$, and $\left|D_{i_1} \setminus M'_{i_1}\right| > \left|D_{i_1} \setminus M_{i_1}\right|$. In this case, $(i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$ may not be a chain from M to M'. We use the pointing procedure from M' to M, which starts with $j_1 = i_1$ pointing to some $c_1 \in D_{i_1}$ such that $c_1 \in M_{i_1} \setminus M'_{i_1}$. Then a cycle or a chain from M to M' can be found.

Case 4. The procedure stops at $i_{\bar{t}}$ at Step $\bar{t}-1$ and $i_{\bar{t}} = i_1$.

Subcase 4.1. $d_{\bar{t}-1} \in D_{i_1}$.

To see that $(i_1, d_1, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M', we verify Condition 2 in the definition of a cycle for i_1. Since $d_{\bar{t}-1} \in D_{i_1}$ and $d_{\bar{t}-1} \in M_{i_1}$, $\left|D_{i_1} \setminus M_{i_1}\right| < \left|D_{i_1}\right|$. Then given that $|M'_{i_1} \setminus D_{i_1}| > |M_{i_1} \setminus D_{i_1}|$, by Assumption 2, there exists $s > \left|D_{i_1} \setminus M_{i_1}\right|$ such that $(|M'_{i_1} \setminus D_{i_1}|, s) \in \mathcal{S}_{i_1}(D_{i_1})$. It follows from Assumption $1'$ that

$$(\left|M_{i_1} \setminus D_{i_1}\right| + 1, \left|D_{i_1} \setminus M_{i_1}\right| + 1) \in \mathcal{S}_{i_1}(D_{i_1}).$$

Similarly, $d_{\bar{t}-1} \in D_{i_1}$ and $d_{\bar{t}-1} \notin M'_{i_1}$ imply that $\left|D_{i_1} \setminus M'_{i_1}\right| > 0$. Then by Assumption 2, there exists $s' < \left|D_{i_1} \setminus M'_{i_1}\right|$ such that $(|M_{i_1} \setminus D_{i_1}|, s') \in \mathcal{S}_{i_1}(D_{i_1})$. It follows from Assumption $1'$ that

$$(\left|M'_{i_1} \setminus D_{i_1}\right| - 1, \left|D_{i_1} \setminus M'_{i_1}\right| - 1) \in \mathcal{S}_{i_1}(D_{i_1}).$$

Subcase 4.2. $d_{\bar{t}-1} \notin D_{i_1}$ and $\beta_{d_{\bar{t}-1}} \in \hat{\mathcal{C}}'(\beta_{i_1})$.

Then $(i_1, d_1, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M'.

Subcase 4.3. $d_{\bar{t}-1} \notin D_{i_1}$ and $\beta_{d_{\bar{t}-1}} \notin \hat{\mathcal{C}}'(\beta_{i_1})$.

Then $(i_1, d_1, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is not a cycle from M to M'. We can still carry out the exchanges in this pseudo-cycle based on M: remove d_1 from M_{i_2} and add it to M_{i_1}, …, remove $d_{\bar{t}-1}$ from M_{i_1} and add it to $M_{i_{\bar{t}-1}}$. This leads to a matching M^1 for $\hat{D}$ and $\hat{\mathcal{C}}$. Since $d_1 \notin D_{i_1}$ and $d_{\bar{t}-1} \notin D_{i_1}$, the schedule of i_1 under M^1 is the same as her schedule under M. Then we can repeat the previous analysis and look for a cycle or a chain from M^1 to M'. Note that this Subcase 4.3 may be reached again. That is, we may find another pseudo-cycle in the pointing procedure from M^1 to M', in which i_1 is pointed by some $d \notin D_{i_1}$ and $\beta_d \notin \hat{\mathcal{C}}'(\beta_{i_1})$.[66] In this case, we can carry out the exchanges in the second pseudo-cycle based on M^1, which leads to a matching M^2 for $\hat{D}$ and $\hat{\mathcal{C}}$. The schedule of i_1 remains the same under M^2 and we look for a cycle or a chain from M^2 to M'. We continue in this fashion. As the set $\{d \in M_{i_1} \setminus D_{i_1} : \beta_d \notin \hat{\mathcal{C}}'(\beta_{i_1})\}$ is finite, after a finite number of steps, some M^k, $k \geq 1$, is constructed and a cycle or a chain C from M^k to M' is found. Using arguments similar to those in the proof of Lemma 3, it can be shown that C is also a cycle or a chain from M to M'. ■

[66] Note that since there can be multiple objects of the same type, there may exist multiple pseudo-cycles even if $|\hat{\mathcal{C}}(\beta_{i_1}) \setminus \hat{\mathcal{C}}'(\beta_{i_1})| = 1$.

Finally, by arguments similar to those in the proof of Lemma 4, we can use Lemma S.8 to show Lemma S.7. This concludes the proof of Theorem S.5.

C Weighted Maximal Mechanisms: Additional Results

C.1 Sequential Targeting Mechanisms are Weighted Maximal

Let $I = \{1, 2, \ldots, |I|\}$ be the set of patients. In this section, for the ease of matrix operations we use a slightly more general definition of an allocation. For every $D \in \mathcal{D}$, $\alpha \in \mathcal{A}(D)$ and $i \in I$, $\alpha_X(i)$ is defined for every blood type $X \in \mathcal{B}$ by setting $\alpha_X(i) = 0$ for all $X \in \mathcal{B} \setminus \mathcal{C}(\beta_i)$.

Let f be a sequential targeting mechanism with respect to target sets $\{N_k\}_{k=1}^{\bar{k}}$ and target function τ. Consider any problem $D \in \mathcal{D}$. For each $k \in \{1, \ldots, \bar{k}\}$, we define a function $W_k : \mathcal{A}(D) \to \mathbb{Z}$ such that for every $\alpha \in \mathcal{A}(D)$,

$$W_k(\alpha) = \begin{cases} \sum_{i \in N_k, X \in \mathcal{B}} \alpha_X(i) & \text{if } \tau(k) = \max \\ -\sum_{d \in \cup_{i \in N_k} D_i} \alpha(d) & \text{if } \tau(k) = \min \end{cases}.$$

Let $h \in \mathbb{Z}_{++}$. Define a function $W : \mathcal{A}(D) \to \mathbb{R}$ such that for every $\alpha \in \mathcal{A}(D)$,

$$W(\alpha) = \sum_{k=1}^{\bar{k}} h^{\bar{k}-k} W_k(\alpha) = \sum_{i \in I} \Big(W^r(i) \cdot \alpha(i) - W^s(i) \cdot \sum_{d \in D_i} \alpha(d) \Big),$$

where

$$W^r(i) = \sum_{k : i \in N_k, \tau(k) = \max} h^{\bar{k}-k}$$

and

$$W^s(i) = \sum_{k : i \in N_k, \tau(k) = \min} h^{\bar{k}-k}.$$

Suppose that $k \in \{1, \ldots, \bar{k} - 1\}$, $\alpha, \alpha' \in \mathcal{A}_{k-1}$ and $W_k(\alpha) > W_k(\alpha')$. Since $W_\ell(\alpha) = W_\ell(\alpha')$ if $\ell < k$, and

$$W_\ell(\alpha') - W_\ell(\alpha) \leq \sum_{X \in \mathcal{B}} \upsilon_X + \sum_{i \in I} \max_{D'_i \in \mathcal{D}_i} |D'_i|$$

if $\ell > k$, we have $W(\alpha) > W(\alpha')$ if

$$h^{\bar{k}-k} > \sum_{\ell=k+1}^{\bar{k}} h^{\bar{k}-\ell} \cdot \Big(\sum_{X \in \mathcal{B}} \upsilon_X + \sum_{i \in I} \max_{D'_i \in \mathcal{D}_i} |D'_i| \Big).$$

This is equivalent to

$$1 > \sum_{\ell=k+1}^{\bar{k}} h^{k-\ell}\Big(\sum_{X\in\mathcal{B}} v_X + \sum_{i\in I} \max_{D_i'\in\mathcal{D}_i} |D_i'|\Big).$$

Therefore, after choosing sufficiently large h such that

$$1 > \sum_{\ell=2}^{\bar{k}} h^{1-\ell}\Big(\sum_{X\in\mathcal{B}} v_X + \sum_{i\in I} \max_{D_i'\in\mathcal{D}_i} |D_i'|\Big), \tag{1}$$

we have for any $k \in \{1,\dots,\bar{k}\}$ and any $\alpha, \alpha' \in \mathcal{A}_{k-1}$, $W_k(\alpha) > W_k(\alpha')$ implies $W(\alpha) > W(\alpha')$. Then, given that all the allocations in $\mathcal{A}_{\bar{k}}$ are welfare equivalent, the sequential targeting outcome $f(D) \in \mathcal{A}_{\bar{k}}$ is welfare equivalent to any solution to the following maximization problem:

$$\max_{\alpha\in\mathcal{A}(D)} W(\alpha)$$

Recall that each patient $i \in I$ first appears in a maximization target: for every $k \in \{2,\dots,\bar{k}\}$, if $\tau(k) = \min$, then for any $i \in N_k$ there exists $k' < k$ such that $i \in N_{k'}$ and $\tau(k') = \max$. This implies that for every $i \in I$, $W^r(i) \geq W^s(i)|D_i|$ for all $D_i \in \mathcal{D}_i$, as h satisfies inequality (1). Therefore, f is a weighted maximal mechanism with respect to the score function with the individual weights $W^r(i)$ and $W^s(i)$. This shows the following proposition.

Proposition 1. *Every sequential targeting mechanism is a weighted maximal mechanism.*

C.2 A Polynomial-time Method for Weighted Maximal Mechanisms

Consider any weighted maximal mechanism f and any problem $D \in \mathcal{D}$. For every allocation $\alpha \in \mathcal{A}(D)$, let

$$\alpha_i = \Big(\big(\alpha_X(i)\big)_{X\in\mathcal{B}}, \big(\alpha(d)\big)_{d\in D_i}\Big) \qquad \text{and} \qquad \alpha = (\alpha_i)_{i\in I} \in \mathbb{Z}_+^a,$$

where the dimension of an allocation is

$$a = |I| \cdot |\mathcal{B}| + \sum_{i\in I} |D_i|.$$

Using a construction similar to the one in Appendix C.1, it can be shown that there exists an $a \times 1$ weight vector W such that $f(D)$ is welfare equivalent to any solution to the following maximization problem

$$\max_{\alpha\in\mathcal{A}(D)} \alpha \cdot W$$

Suppose that Assumption 1 (L-convexity) holds. Given $\alpha \in \mathbb{Z}_+^a$, we show that the constraint "α is an allocation", i.e., $\alpha \in \mathcal{A}(D)$, is equivalent to a system of linear inequalities in four parts:

1. For every patient $i \in I$, let

$$r_i = \sum_{X \in \mathcal{B}} \alpha_X(i) \qquad \text{and} \qquad s_i = \sum_{d \in D_i} \alpha(d).$$

Since $\mathcal{S}_i(D_i)$ is L-convex, there exists some integer vector $b_i \in \mathbb{Z}^6$ such that $(r_i, s_i) \in \mathcal{S}_i(D_i)$ if and only if the following inequalities hold:

$$\begin{aligned} r_i - s_i &\le b_{i,1} \\ -r_i + s_i &\le b_{i,2} \\ r_i &\le b_{i,3} \\ -r_i &\le b_{i,4} \\ s_i &\le b_{i,5} \\ -s_i &\le b_{i,6} \end{aligned}$$

We rewrite these linear inequalities in matrix form, after defining

$$A_i = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 1 & 0 & 0 & 1 & -1 \end{pmatrix} \qquad \forall\, i \in I, \tag{2}$$

$$A_I = \begin{pmatrix} A_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & A_{|I|} \end{pmatrix}, \qquad \text{and} \qquad b_I = (b_i)_{i \in I},$$

as follows:

$$\alpha \cdot A_I \le b_I. \tag{3}$$

Note that the minimum guarantees and the maximum needs are handled through these inequalities.

2. We rewrite the market clearing conditions,

$$\sum_{i \in I : X \in \mathcal{C}(\beta_i)} \alpha_X(i) - \sum_{d \in \cup_{i \in I} D_i : \beta_d = X} \alpha(d) \le v_X \qquad \forall\, X \in \mathcal{B},$$

in matrix inequality form as

$$\alpha \cdot A_{\mathcal{B}} \le v \tag{4}$$

where

$$A_{\mathcal{B}} = (A_X^T)_{X \in \mathcal{B}}$$

defined by $\forall\, X \in \mathcal{B}$,

$$A_X = \Big(\big(A_X(i,Y)\big)_{Y\in\mathcal{B}}, \big(A_X(d)\big)_{d\in D_i}\Big)_{i\in I}$$

such that

$$A_X(i,Y) = \begin{cases} 1 & \text{if } Y = X \text{ and } X \in \mathcal{C}(\beta_i) \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, i \in I, \;\; \forall\, Y \in \mathcal{B}$$

and

$$A_X(d) = \begin{cases} -1 & \text{if } \beta_d = X \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, d \in \cup_{i\in I} D_i.$$

3. The following inequality states that a donor never exceeds 1 unit of donation:

$$\alpha(d) \le 1 \qquad \forall\, d \in \cup_{i\in I} D_i.$$

   We rewrite this as

$$\alpha \cdot A_D \le b_D = (1,\ldots,1) \tag{5}$$

   where

$$A_D = \big(A_D(r,c)\big)_{r\le a,\; c\le \cup_{i\in I}|D_i|}$$

   such that $A_D(r,c) = 1$ if both row r and column c refer to the same donor d, and $A_D(r,c) = 0$ otherwise.

4. Finally, no patient receives incompatible blood:

$$\sum_{i\in I} \sum_{X\in\mathcal{B}\setminus\mathcal{C}(\beta_i)} \alpha_X(i) \le 0,$$

   which can be written as

$$\alpha \cdot A_{\mathcal{C}} \le 0 \tag{6}$$

   where

$$A_{\mathcal{C}} = \left(\Big(\big(A_{\mathcal{C}}(i,X)\big)_{X\in\mathcal{B}}, \big(A_{\mathcal{C}}(d)\big)_{d\in D_i}\Big)_{i\in I}\right)^T$$

   such that

$$A_{\mathcal{C}}(i,X) = \begin{cases} 1 & \text{if } X \notin \mathcal{C}(\beta_i) \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, i \in I, \;\; \forall\, X \in \mathcal{B}$$

   and

$$A_{\mathcal{C}}(d) = 0 \qquad \forall\, d \in \cup_{i\in I} D_i.$$

Then the vector $\alpha \in \mathbb{Z}^a_+$ is an allocation, i.e., $\alpha \in \mathcal{A}(D)$, if and only if inequalities (3), (4), (5), and (6) hold. This implies that the following integer linear program in cannonical form finds an allocation that is welfare equivalent to $f(D)$:

$$\max\ \alpha \cdot W \tag{7}$$

subject to

$$\alpha \cdot A \leq b \tag{8}$$

where

$$A = (A_I, A_{\mathcal{B}}, A_D, A_{\mathcal{C}}) \qquad \text{and} \qquad b = (b_I, v, b_D, 0)$$

such that α is a $1 \times a$ non-negative integer vector, A is an $a \times (6|I| + |\mathcal{B}| + |\cup_{i\in I} D_i| + 1)$ integer matrix with entries 0, 1 or -1, and b is a $1 \times (6|I| + |\mathcal{B}| + |\cup_{i\in I} D_i| + 1)$ integer vector. We consider its linear program relaxation such that the search space is $\mathbb{R}^a_+$ instead of $\mathbb{Z}^a_+$.

A matrix is **totally unimodular** if the determinant of every square submatrix is $-1, 0$ or 1. The following result is well known and straightforward to prove using Cramer's rule in linear algebra (for example, see Schrijver (1998)).

Lemma S.9. *The vertices of the polyhedron defined by the inequality (8) are integer-valued for any integer vector b if and only if A is totally unimodular.*

Thus, for any linearly independent basis for α the linear program relaxation of the problem in (7) and (8) has only integer solutions for any integer vector b if and only if A is totally unimodular. The following lemma establishes a condition for checking the total unimodularity of A:

Lemma S.10 (Ghouila-Houri (1962))**.** *A is totally unimodular if and only if there exists a partition of any subset of column indices $C \subseteq \{1, 2, \ldots, 6|I| + |\mathcal{B}| + |\cup_{i\in I} D_i| + 1\}$ as K_C and L_C such that for the column vector $\kappa = \sum_{c\in K_C} A^c - \sum_{c\in L_C} A^c$, where A^c is the c^{th} column vector of A, we have $\kappa(r) \in \{-1, 0, 1\}$ for every row $r = 1, \ldots, a$.*

We prove that A is indeed totally unimodular using this result.

Lemma S.11. *The matrix A is totally unimodular.*

Proof of Lemma S.11. Let $C \subseteq \{1, 2, \ldots, 6|I| + |\mathcal{B}| + |\cup_{i\in I} D_i| + 1\}$ be any subset of column indices of A. We construct a partition of C, K_C and L_C, as in Lemma S.10 in four steps. Below for each $x \in \{1, 2, 3, 4\}$, κ^x denotes the difference vector between the sum of the columns with indices in K_C and the sum of the columns with indices in L_C at the end of the construction in Step x.

1. We first consider the columns that correspond to the feasible schedule constraints. Let $i \in I$. List the column indices in the set $\{c \in C : 6(i-1) + 1 \leq c \leq 6i\}$

as $c_1, c_2, \ldots, c_k$. We will inductively assign these indices to two sets, K_C^i and L_C^i, which are both initialized to $\emptyset$. Let the index of the first row regarding i in each column be

$$r = (i-1)|\mathcal{B}| + \sum_{j<i} |D_j| + 1,$$

and the index of the first row regarding i's donors be

$$r' = i|\mathcal{B}| + \sum_{j<i} |D_j| + 1.$$

For every ℓ such that $1 \leq \ell \leq k+1$, let $\kappa_{\ell-1}$ denote the difference vector between the sum of the columns with indices in K_C^i and the sum of the columns with indices in L_C^i, after every column $c_{\ell'}$ with $\ell' < \ell$ is assigned to either K_C^i or L_C^i. We keep track of the two entries $\kappa_{\ell-1}(r)$ and $\kappa_{\ell-1}(r')$, since by the construction of A_i, the entries at other rows of $\kappa_{\ell-1}$ regarding blood received by i are identical to $\kappa_{\ell-1}(r)$ and the entries at other rows of $\kappa_{\ell-1}$ regarding the donors of i are identical to $\kappa_{\ell-1}(r')$. In the construction we will keep the two entries $\kappa_{\ell-1}(r)$ and $\kappa_{\ell-1}(r')$ be -1, 0 or 1, and make sure that they do *not have the same sign*.

Fix ℓ with $1 \leq \ell \leq k$. Assume as the inductive assumption that the vector ζ, defined as $\zeta = (\kappa_{\ell-1}(r), \kappa_{\ell-1}(r'))$, has entries -1, 0, or 1 and its two entries do not have the same sign (the initial case is covered as for $\ell = 1$, $\zeta = (0, 0)$).

Consider column c_ℓ and let $\chi = (A(r, c_\ell), A(r', c_\ell))$. Note that χ's entries do not have the same sign by the construction of A_i. Moreover, if we assign c_ℓ to K_C^i, then the relevant difference of sums will be κ_ℓ such that

$$(\kappa_\ell(r), \kappa_\ell(r')) = \zeta + \chi,$$

and if we assign c_ℓ to L_C^i, then

$$(\kappa_\ell(r), \kappa_\ell(r')) = \zeta - \chi.$$

The following four cases cover all the possibilities (noting that $\chi \neq (0, 0)$ by the construction of A_i):

- If $\zeta = (0, 0)$, assign c_ℓ to K_C^i. Then $(\kappa_\ell(r), \kappa_\ell(r'))$ satisfies the inductive claim as χ's entries do not have the same sign.
- If exactly one of $\zeta(1)$ and $\chi(1)$ is 0 and exactly one of $\zeta(2)$ and $\chi(2)$ is 0: Then suppose $\chi(m)$ and $\zeta(n)$ are nonzero for $m \neq n$. If they have the same sign, then assign c_ℓ to L_C^i. If they have opposite signs, then assign c_ℓ to K_C^i. Thus,

$$(\kappa_\ell(r), \kappa_\ell(r')) = (-x, x)$$

where $x \in \{-1, 1\}$.

- If $\zeta(1)$ and $\chi(1)$ have the same sign, then assign c_ℓ to L^i_C; and if they have opposite signs, then assign c_ℓ to K^i_C. In the former case $\zeta(2)$ and $\chi(2)$ cannot have opposite signs and in the latter case $\zeta(2)$ and $\chi(2)$ cannot have the same sign, by the inductive assumption and the construction of A_i. Thus,
$$(\kappa_\ell(r), \kappa_\ell(r')) = (0, x)$$
where $x \in \{-1, 0, 1\}$.
- If $\zeta(2)$ and $\chi(2)$ have the same sign, then assign c_ℓ to L^i_C; and if they have opposite signs, then assign c_ℓ to K^i_C. Similarly,
$$(\kappa_\ell(r), \kappa_\ell(r')) = (x, 0)$$
where $x \in \{-1, 0, 1\}$.

Thus, it has been shown that $(\kappa_\ell(r), \kappa_\ell(r'))$ also satisfies the inductive claim. We repeat the above procedure. After c_k is assigned, the sets K^i_C and L^i_C are constructed. Then we assign the elements in these two sets to K_C or L_C as follows:

- If $\kappa_k(r) = 1$, then assign all column indices in K^i_C to K_C and all column indices in L^i_C to L_C (i.e., keep their orientation).
- If $\kappa_k(r) = -1$, then assign all columns indices in K^i_C to L_C and all column indices in L^i_C to K_C (i.e., reverse their orientation).
- If $\kappa_k(r) = 0$ and $\kappa_k(r') \in \{-1, 0\}$, then assign all column indices in K^i_C to K_C and all column indices in L^i_C to L_C; if $\kappa_k(r) = 0$ and $\kappa_k(r') = 1$, then assign all column indices in K^i_C to L_C and all column indices in L^i_C to K_C.

By this operation after each patient $i \in I$ is handled, κ^1 is defined. The entry in every row of κ^1 regarding a patient's received blood is either 0 or 1, and the entry in every row of κ^1 regarding a donor is either 0 or -1.

2. Consider the columns of $A_{\mathcal{B}}$ and list the column indices in the set $\{c \in C : 6|I| < c \leq 6|I| + |\mathcal{B}|\}$ as $c_1, c_2, \ldots, c_k$. Each of them refers to the market clearing condition for some blood type. Note that for any row r regarding a patient's received blood,
$$\sum_{\ell=1}^{k} A(r, c_\ell) \in \{0, 1\}.$$
On the other hand, for any row r' regarding a donor,
$$\sum_{\ell=1}^{k} A(r', c_\ell) \in \{-1, 0\}.$$
Assign every c_ℓ to L_C. Then we have
$$\kappa^2 = \kappa^1 - \sum_{\ell=1}^{k} A^{c_\ell},$$

which is the difference vector between the sum of the columns with indices in K_C and the sum of the columns with indices in L_C at the end of Step 2.

For the row r defined above we have $\kappa^2(r) \in \{-1, 0, 1\}$ as $\kappa^1(r) \in \{0, 1\}$. For the row r' defined above we have $\kappa^2(r') \in \{-1, 0, 1\}$ as $\kappa^1(r) \in \{-1, 0\}$.

3. For any $c \in C$ with $6|I| + |\mathcal{B}| < c < 6|I| + |\mathcal{B}| + |\cup_{i \in I} D_i| + 1$, column c is in A_D and refers to some donor with a row number r. Assign c to L_C if $\kappa^2(r) \in \{0, 1\}$, and assign it to K_C otherwise. After all such column indices are assigned, $\kappa^3(r) \in \{-1, 0, 1\}$ for any row r regarding a donor, and $\kappa^3(r') = \kappa^2(r') \in \{-1, 0, 1\}$ for any other row r'.
4. The last column of A is the vector $A_{\mathcal{C}}$ and $A_{\mathcal{C}}(r) \in \{0, 1\}$ for every row r. Consider any row r such that $A_{\mathcal{C}}(r) = 1$. This refers to a patient i and a blood type X such that $X \notin \mathcal{C}(\beta_i)$. Then for any $c \in C$ assigned in Steps 2 and 3, $A(r, c) = 0$. Therefore, $\kappa^3(r) = \kappa^1(r) \in \{0, 1\}$. If the index $6|I| + |\mathcal{B}| + |\cup_{i \in I} D_i| + 1 \in C$, we assign it to L_C so that $\kappa^4(r) \in \{-1, 0\}$. For any row r' such that $A_{\mathcal{C}}(r') = 0$, $\kappa^4(r') = \kappa^3(r') \in \{-1, 0, 1\}$.

Therefore, we have constructed a partition of C, K_C and L_C, such that $\sum_{c \in K_C} A^c - \sum_{c \in L_C} A^c = \kappa^4$ and $\kappa^4(r) \in \{-1, 0, 1\}$ for every row $r = 1, \ldots, a$. By Lemma S.10, A is totally unimodular. ■

These results are used to prove the following proposition.

Proposition 2. *Under Assumption 1, the outcome of a weighted maximal mechanism can be found in polynomial time.*

Proof of Proposition 2. By Lemmata S.9 and S.11, under Assumption 1, all the basic solutions to the linear program relaxation of the integer linear program in (7) with constraint (8) are integer-valued. Thus, any polynomial LP method, such as the simplex algorithm, finds an allocation that is welfare equivalent to $f(D)$ in polynomial time. ■

D Examples Regarding Violations of Assumptions

Example S.3 and Example S.4 below show that Assumption 1 and Assumption 2 are needed for the donor monotonicity of the optimal mechanisms, respectively.

Example S.3 (Violation of Assumption 1)**.** Suppose that the set of patients is $I = \{1, 2, 3, 4\}$. For every $i \in I$, $\underline{n}_i = 0$. Each patient's blood type, maximum need and donor set are given as follows.

- $\beta_1 = A$, $\overline{n}_1 = 2$, and Patient 1 has two type B donors and four type O donors.

- $\beta_2 = B$, $\overline{n}_2 = 2$, and Patient 2 has four type O donors.
- $\beta_3 = O$, $\overline{n}_3 = 4$, and Patient 3 has one type A donor and seven type AB donors.
- $\beta_4 = A$, $\overline{n}_4 = 1$, and Patient 4 has two type AB donors.

In addition, the blood bank only has one unit of type A blood in its inventory. Assume ABO-identical transfusion.

For every $i \in I$ and every possible donor set $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \Big\{(r, s) \in \mathbb{W}_i : s = 2r \text{ and } r \leq \min\big\{\overline{n}_i, \big\lfloor |D_i|/2 \big\rfloor\big\}\Big\}.$$

Note that Assumptions 2 and 3 are satisfied, while Assumption 1 is violated: if a patient reports at least two donors, then her feasible schedule set is not L-convex.

Let f be a sequential targeting mechanism with respect to target sets $\{N_k\}_{k=1}^{\bar{k}}$ and target function τ such that $N_1 = N_2 = \{3\}$, and $N_3 = N_4 = \{4\}$. Then f selects the following allocation when every patient truthfully reports her donor set:

- Patient 1 receives one unit of type A blood and her two type B donors donate.
- Each $i \in \{2, 3, 4\}$ receives $\overline{n}_i$ units of type β_i blood and all the donors of i donate.

If Patient 1 conceals her two type B donors, then f selects the following allocation:

- Patient 1 receives two units of type A blood and her four type O donors donate.
- Patient 3 receives four units of type O blood and all of her donors donate.
- Patient 2 and Patient 4 do not receive any blood and their donors do not donate.

Therefore, Patient 1 successfully manipulates. In the original problem both Patient 1 and Patient 2 can provide type O blood to the patient with the highest priority, Patient 3. After Patient 1 conceals her type B donors, Patient 2 cannot receive or supply any blood, and hence Patient 1's four type O donors donate. Then the two-for-one exchange rate requires Patient 1 to receive the two units of type A blood, despite that the other type A Patient, Patient 4, has higher priority. ∎

Example S.4 (Violation of Assumption 2)**.** Suppose that the set of patients is $I = \{1, 2, 3, 4\}$. For every $i \in I$, $\underline{n}_i = 0$. Let $\overline{n}_1 = 2$, and $\overline{n}_i = 1$ for every $i \in I \setminus \{1\}$. Each patient's blood type and donor set are given as follows.

- $\beta_1 = A$, and Patient 1 has one type B donor and one type O donor.
- $\beta_2 = B$, and Patient 2 has one type AB donor.
- $\beta_3 = AB$, and Patient 3 has one type A donor and one type O donor.
- $\beta_4 = O$, and Patient 4 has one type A donor.

In addition, the blood bank only has one unit of type AB blood in its inventory. Assume ABO-identical transfusion.

The exchange rate is one-for-one for every $i \in I \setminus \{1\}$. That is, for every reported donor set $D_i \in \mathcal{D}_i$, where $i \in I \setminus \{1\}$,

$$\mathcal{S}_i(D_i) = \begin{cases} \{(0,0)\} & \text{if } D_i = \emptyset \\ \{(0,0),(1,1)\} & \text{otherwise} \end{cases}.$$

On the other hand, Patient 1 can receive blood up to her maximum need by supplying at most one unit: for every $D_1 \in \mathcal{D}_1$,

$$\mathcal{S}_1(D_1) = \begin{cases} \{(0,0)\} & \text{if } D_1 = \emptyset \\ \{(0,0),(1,0),(1,1),(2,0),(2,1)\} & \text{otherwise} \end{cases}.$$

This is a special case of the Delhi policy in Example 1. Note that Assumptions 1 and 3 are satisfied. However, Assumption 2 is violated, since when Patient 1 reports two donors, $(2,2)$ is not a feasible schedule.

Let f be a sequential targeting mechanism with respect to target sets $\{N_k\}_{k=1}^{\bar{k}}$ and target function τ such that $N_1 = \{2\}$. Then f selects the following allocation when every patient truthfully reports her donor set:

- Each $i \in I$ receives one unit of type β_i blood.[67]
- Patient 1's type B donor donates, Patient 3's type O donor donates, and the donor of $i \in \{2,4\}$ donates.

If Patient 1 conceals her type B donor, then f selects the following allocation:

- Patient 1 receives two units of type A blood and her type O donor donates.
- Patient 2 receives nothing and her donor does not donate.
- Patient 3 receives one unit of type AB blood and her type A donor donates.
- Patient 4 receives one unit of type O blood and her type A donor donates.

Therefore, Patient 1 successfully manipulates. ▪

E Model Extensions for Policy Design

E.1 Preferences of Blood Bank and Inventory Objectives

After collecting blood from the replacement donors and distributing blood to the patients, the blood bank may have some remaining blood of each type in its inventory. It can have preferences over different remaining inventories and such preferences can correspond to some explicit objectives, such as maximizing the amount of certain types of blood in stock. To this end, we extend our model and include the blood bank b as an agent. In an allocation α, we also specify the amount of type X blood the bank receives, $\alpha_X(b)$, for each $X \in \mathcal{B}$. Denote a *blood bundle* that the bank keeps in its inventory

[67]Note that every patient's need would be fully satisfied if $(2,2)$ were a feasible schedule for Patient 1.

as $z = (z_X)_{X \in \mathcal{B}} \in \mathbb{Z}_+^{|\mathcal{B}|}$. Assume that the bank has a complete preference relation over all the blood bundles. Then the definition of efficiency can be modified accordingly to include the bank's welfare. A schedule profile is extended and denoted by a vector $w = \big((r_i, s_i)_{i \in I}, (z_X)_{X \in \mathcal{B}}\big) \in \mathbb{W} \times \mathbb{Z}_+^{|\mathcal{B}|}$. The mechanism designer's preference relation $\succeq$ over all such schedule profiles is complete, transitive, antisymmetric, and responsive to the basic schedule profiles in the set $\{0, 1\}^{2|I|+|\mathcal{B}|}$. Moreover, $\succeq$ is aligned with the preferences of all the agents (all the patients and the bank): for every two schedule profiles w and w', we have $w \succeq w'$ if every agent weakly prefers w to w', and at least one agent strictly prefers w to w'.[68] Then, the optimal mechanism induced by $\succeq$ is efficient, and it is straightforward to extend the proofs to show that Theorem 2 and Theorem 3 remain valid.

We give a simple and concrete example of an optimal mechanism in this more general environment. Consider a sequential targeting mechanism with respect to target sets $\{N_k\}_{k=1}^{\bar{k}}$ and target function τ. We first add $|\mathcal{B}|$ singleton target sets to the end of the sequence as additional tie breakers: each of them only includes b, and for each $X \in \mathcal{B}$, the target of maximizing the amount of type X blood received by the bank is assigned to one of them. Suppose that the bank is mainly concerned with the total amount of blood in stock, as well as its inventory of the rare Rh D− blood, and its preferences are as follows: for any two blood bundles, it prefers the one with a larger total amount of blood; if the total amounts are the same, it prefers the one with a larger amount of Rh D− blood. To incorporate such preferences or inventory objectives into the sequential targeting mechanism, we can add two additional target sets, such that in the extended sequence $\{\bar{N}_k\}_{k=1}^{\bar{k}+|\mathcal{B}|+2}$, for some k and ℓ we have $k < \ell \leq \bar{k} + 2$, $\bar{N}_k = \bar{N}_\ell = \{b\}$, the target for $\bar{N}_k$ is to maximize the total amount of blood received by the bank, and the target for $\bar{N}_\ell$ is to maximize the amount of Rh D− blood received by the bank. Such an extended sequential targeting mechanism is induced by an underlying preference relation of the designer that is complete, transitive, antisymmetric, and responsive. It is also aligned with every agent's preferences. In particular, the specification of the target sets $\bar{N}_k$ and $\bar{N}_\ell$ ensures that it is aligned with the bank's preferences.[69]

[68] Note that the bank can have a weak preference relation. Hence, it is possible that every agent is indifferent between two different schedule profiles. In this case, the designer must strictly prefer one to the other.

[69] Such preference alignment can generally be achieved in multiple ways. For instance, we can also add only one singleton target set that includes the bank, instead of two, before the $|\mathcal{B}|$ tie breakers, and the corresponding target is to maximize the bank's preferences.

E.2 Integrated Blood Component Markets

Although in practice replacement donor programs function for each blood component separately, it is plausible that higher welfare gains can be achieved by integrating these markets. For instance, a patient requesting red blood cells can have her donors donate platelets to another patient, while the latter patient's donors donate red blood cells to the former patient. The gains from an integrated market do not only come from the increase in market size or number of goods being exchanged, but also the differences in blood type compatibility requirements for different components. For example, a relatively lenient transfusion policy may require ABO-cellular compatible and Rh D compatible red blood cell transfusion, and ABO-plasma compatible platelet transfusion. Then, while a type $AB+$ donor can only donate to a type $AB+$ patient in a separated red blood cell market, she becomes a universal donor in the platelet market. If there is no $AB+$ patient in the red blood cell market, this donor will not be utilized in a separated market. On the other hand, her paired patient will likely receive more blood by utilizing her through exchanges in the integrated market, even if there is no $AB+$ patient in the platelet market. Similarly, a type $O-$ donor of a patient requesting platelets can be utilized more efficiently in the integrated market.

Compared to red blood cells, platelets, and whole blood, plasma shortages are less frequent. This is mainly due to its much longer shelf life. Moreover, red blood cells are in the highest demand and their large market also contributes to the large supply of plasma for transfusion, since each unit of red blood cells has to be prepared from one unit of donated whole blood that also produces one unit of plasma at the same time.[70] Thus, replacement donor programs carry extra importance for red blood cells, platelets, and whole blood.

We can slightly modify our baseline model to integrate the red blood cell (denoted as rbc), platelet (denoted as plt), and whole blood (denoted as wb) markets. As in Appendix B, we interpret $\mathcal{B}$ as the set of *types* of patients and donors, instead of blood types. First, the type of each patient i, β_i, is extended to specify which component she needs. Hence,

$$\mathcal{B}^I = \big\{(c, X) \ : \ c \in \{rbc, plt, wb\} \text{ and } X \in \{O+, O-, A+, A-, B+, B-, AB+, AB-\}\big\}$$

is the set of patient types. We assume that each donor can donate either one unit of apheresis platelets, or one unit of whole blood, which can simply be used as a whole blood transfusion pack, or to prepare one unit of red blood cells. Therefore, each donor

[70] However, other plasma products, including convalescent plasma and source plasma (see Section 2.1), are often in short supply. In the design of replacement donor programs, we only focus on the plasma used in everyday transfusion.

d can provide 1 unit of rbc, 1 unit of plt, or 1 unit of wb. We use the pair (d, X), where X is her blood type, to denote her type β_d. Then

$$\mathcal{B}^D = \big\{(d, X) : d \in \cup_{i\in I, D_i \in \mathcal{D}_i} D_i, X \in \{O+, O-, A+, A-, B+, B-, AB+, AB-\}\big\}$$

is the set of all possible donor types, and $\mathcal{B} = \mathcal{B}^I \cup \mathcal{B}^D$.

Assume that $v_{(d,X)} = 0$ for every $(d, X) \in \mathcal{B}^D$. For each $(c, X) \in \mathcal{B}^I$, the blood bank has $v_{(c,X)}$ units of component c of blood type X. Moreover, the set of compatible types $\mathcal{C}(c, X)$ is defined according to the blood type compatibility requirement for the component c.[71] The other elements of the model as well as the mechanisms are defined as before, and the main results of the paper remain valid for the integrated blood component allocation. Note that, due to the specification of donor types, an allocation specifies the kind of component donated by a replacement donor: If $\alpha_{\beta_d}(i) = 1$, then the donor d provides one unit of the component requested by patient i. Finally, for the integrated market, we can also take into account the kind of blood component requested by each patient in the design of feasible schedule correspondences and optimal mechanisms.

[71]For instance, under ABO-plasma compatible platelet transfusion, $\mathcal{C}(plt, A+) = \big\{(plt, X) : X \in \{A+, A-, AB+, AB-\}\big\} \cup \big\{(d, X) : X \in \{A+, A-, AB+, AB-\}\big\}$.