Public Goods, Social Alternatives, and the Lindahl-VCG Relationship*

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Abstract

Lindahl prices, set by a fictitious auctioneer with full knowledge of values and costs, are a generalization of Walrasian prices. By making the efficient allocation utility- and profit-maximizing for all participants, they induce an efficient outcome in a decentralized way even in the presence of public goods. We study a collective choice model with quasilinear utility, which encompasses the allocation of public and private goods as special cases. We show that each agent's smallest Lindahl price for the efficient alternative is equal to his VCG transfer while the firm's VCG transfer is equal to the largest sum of Lindahl prices. Thus, the VCG mechanism incurs a deficit if and only if the set of vectors of the agents' Lindahl prices for the efficient alternative is multi-valued. Unlike Walrasian prices, Lindahl prices are not restricted to be anonymous or linear. This is the reason why, when considering the allocation of private goods, the agents' smallest Walrasian payments are at least as large as their smallest Lindahl prices, and thus their VCG transfers. It is also why Lindahl prices always exist while Walrasian prices may not.

Keywords: collective choice; Lindahl prices; mechanism design; private goods; public goods; VCG transfers.

JEL-Classification: C72; D44; D61

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1 Introduction

Achieving an efficient provision of public goods is complicated by the agents' incentive to free ride (Hume, 1739; Wicksell, 1896; Samuelson, 1954). In settings with complete information, Lindahl prices (Lindahl, 1919) induce agents to demand and the profit-maximizing firm to supply the efficient level of a public good without requiring outside funds. However, Lindahl prices are often dismissed on the grounds of being impractical because their implementation requires a firm with market power to behave as a price taker and the ability to exclude those agents who do not pay (Mas-Colell et al., 1995, p. 364). Furthermore, Lindahl prices are not constructed to provide the participants with the correct incentives to reveal the information about their preferences truthfully. As a case in point, Samuelson (1954, p. 388/9) notes:

> But, and this is the point sensed by Wicksell but perhaps not fully appreciated by Lindahl, now it is in the selfish interest of each person to give false signals, to pretend to have less interest in a given collective consumption activity than he really has, etc.

In contrast, with quasilinear utility, the seemingly unrelated VCG mechanism, which is designed to explicitly account for the participants' private information, provides them with dominant strategies to report their preferences truthfully and induces efficient production while respecting the participants' individual rationality constraints. However, it typically runs a deficit.[1]

In this paper, we show that there is a tight connection between Lindahl prices and VCG transfers for the following general collective choice problem. There are finitely many mutually exclusive social alternatives, which a firm can produce at some cost, and agents with quasilinear utility. This setup encompasses a wide range of alternative interpretations. For example, the social alternatives may represent different or differently sized public goods or correspond to different allocations of finitely many private goods among the agents.

[1] The VCG mechanism derives its label from the independent contributions of Vickrey (1961), Clarke (1971) and Groves (1973). Notable subsequent contributions include Green and Laffont (1977a,b). For different specifications and notions of incentive compatibility, the result that efficient public good provision is not possible without running a deficit has been shown in the literature; see, for example, Güth and Hellwig (1986), Mailath and Postlewaite (1990) and Loertscher and Mezzetti (2019).

A Lindahl price vector is a collection of prices, with each agent being offered an individualized price for every social alternative. A Lindahl price vector must induce every agent to choose the same efficient alternative and the profit-maximizing firm to produce that alternative. In this sense, Lindahl prices implement the efficient allocation in a "decentralized" way while always balancing the budget, since the sum of agents' payments is equal to the firm's revenue.

We first show that every Lindahl price vector supports every efficient allocation, and that the set of Lindahl price vectors is non-empty and compact. Calling the price an agent pays for an efficient allocation his *effective* Lindahl price, we then demonstrate that each agent's VCG transfer is the effective Lindahl price that is most favorable to this agent, that is, his smallest effective Lindahl price. Conversely, the firm's VCG transfer corresponds to the sum of the agents' Lindahl prices (i.e., the firm's revenue) at its preferred effective Lindahl price vector, that is, the maximum revenue the firm obtains from any Lindahl price vector for an efficient allocation.

A consequence of this equivalence result is that the VCG mechanism balances the budget (i.e., the deficit is zero) if and only if there is a unique vector of effective Lindahl prices; that is, a unique vector of Lindahl prices for an efficient allocation.[2] We also characterize the (weaker) conditions under which the sum of the agents' VCG transfers covers the firm's cost of production. This is, for example, the case with a homogeneous private good and agents with single-unit demand.

The study of public good problems has a long tradition in economics. Indeed, what has become known as information economics has its roots in public good problems, originating with the observations of Samuelson (1954) and the subsequent contributions by, for example, Clarke (1971) and Green and Laffont (1977a,b). For a collective choice problem that encompasses private and public good problems as special cases, this paper establishes a general equivalence between the transfers under the VCG mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) and extremal Lindahl prices (Lindahl, 1919; Mas-Colell et al., 1995). This equivalence allows us to revisit results in the literature that relate VCG transfers to Wal-

[2] As we show in Appendix B, this is related to whether what Segal and Whinston (2016) define as the marginal core is single- or multi-valued.

rasian prices. For private goods allocation problems and single-unit demands, or additive payoffs, it is well known that the smallest Walrasian price is the VCG transfer buyers pay; see, for example, Leonard (1983), Demange (1982), and Gul and Stacchetti (1999). Thus, in such cases the smallest Walrasian price is every agent's preferred effective Lindahl price. However, more generally, each agent's payment based on the smallest Walrasian price (if Walrasian prices exist) is an upper bound of his smallest VCG transfer (Gul and Stacchetti, 1999; Delacrétaz et al., 2022). The reason why the equivalence breaks down is that Walrasian prices impose restrictions – linearity and anonymity – that are not part of the VCG mechanism and of Lindahl prices. Since Lindahl prices always exist, these restrictions are also the reason why Walrasian prices may not exist. We show that it is sufficient to add one of these restrictions to break the equivalence between VCG transfers and "restricted" Lindahl prices. Our paper is also related to Gul and Pesendorfer (2022), who provide a cooperative game theory foundation for Lindahl equilibrium based on weighted Nash bargaining, as well as an axiomatization. Our paper complements theirs by highlighting the connection between Lindahl prices and the strategic, mechanism design, notion associated with VCG transfers.

The remainder of the paper is organized as follows. Section 2 provides a motivating and illustrating example. Section 3 introduces the model and the definitions of Lindahl prices and of the VCG mechanism. In Section 4, we establish the equivalence between extremal Lindahl prices and VCG transfers. Price restrictions and their implications are analyzed in Section 5. Section 6 analyzes the conditions under which the VCG mechanism runs no deficit and conditions for cost recovery to be possible. Section 7 concludes. All proofs are in Appendix A. Appendix B connects effective Lindahl price vectors with the marginal core defined by Segal and Whinston (2016).

2 Motivating Example

To illustrate the connection between VCG transfers and Lindahl prices, we begin with an example in which there are two agents, Paul and William, who are interested in reading a paper that Erik, the firm, can write. Erik's cost of writing the paper is 4 while it generates a value of 3 for Paul and a value of 2 for William. The value of the outside option of there being no paper is 0 for both agents and for the firm. Because the welfare when the paper is written – that is, the sum of the agents' values minus the firm's cost – is 1, it is efficient to produce the paper.

As is well known, the VCG transfer an agent pays is the joint surplus the other agent and the firm would obtain without that agent present, or equivalently, if that agent's value were zero, minus the joint surplus that they obtain with that agent present. Because each agent's value is less than the firm's cost, each agent is pivotal for production, meaning that the joint surplus of the firm and the other agent is zero in the absence of this agent. Consequently, each agent is charged a positive VCG transfer. For Paul, this transfer is $2 = -(2-4)$ while for William it is $1 = -(3-4)$. Analogously, the firm's VCG transfer is the agents' joint surplus with the firm present, which is 5, minus their joint surplus without the firm, which is 0. Consequently, the firm's VCG transfer is 5. Observe also that the agents' VCG transfers can equivalently be characterized as *thresholds* payments that are equal to the smallest values each of them could have reported without inducing the allocation to change while the firm's threshold payment is the highest cost it could have reported without inducing the allocation to change.

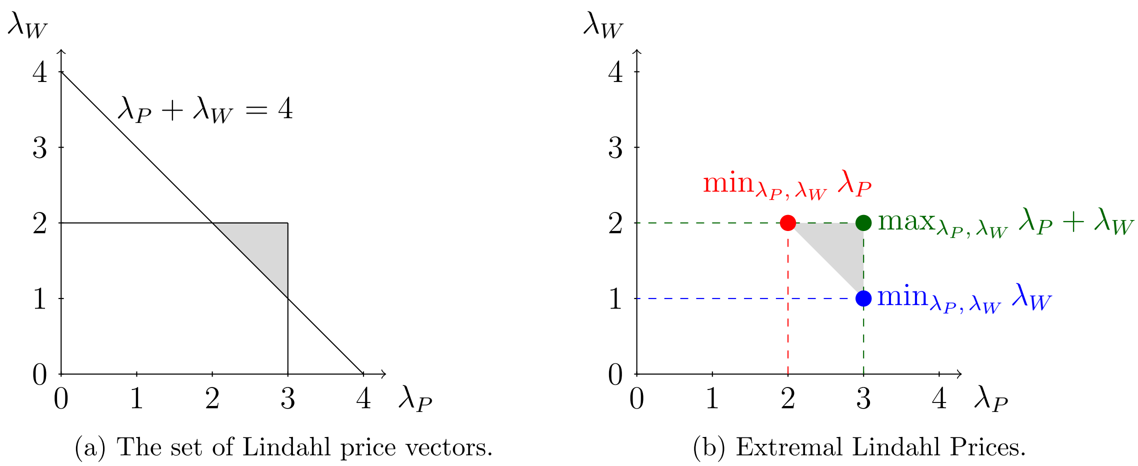

Figure 1: Panel (a): The set of Lindahl price vectors (shaded). Panel (b): Extremal Lindahl prices.

Consider now the set of Lindahl price vectors, which is the set of prices that make choosing the efficient alternative optimal for each agent and for the firm. This set is depicted in Figure 1. Because the value of the outside option is 0, the price for every agent and for the firm for no paper is 0. To induce the efficient choices, a Lindahl price vector (λ_P, λ_W) that charges Paul a price of λ_P and William a price of λ_W has to make it optimal for the agents to buy the paper, which requires $\lambda_P \leq 3$ and $\lambda_W \leq 2$, and for the firm to produce it, which in turn requires $\lambda_P + \lambda_W \geq 4$.

Observe then that the largest sum of Lindahl prices, given by the top right corner of the shaded triangle in Figure 1, is equal to 5, which is precisely the firm's VCG transfer. Likewise, the smallest Lindahl price for Paul, given by the first coordinate of the top left corner of the triangle, is 2, and the smallest Lindahl price for William, corresponding to the second coordinate in the bottom right corner of the triangle, is 1. Thus, each agent's smallest Lindahl price is his VCG transfer. To develop intuition for the equivalence, consider first the firm's problem. The largest sum of Lindahl prices simply consists of the sum of the agents' values. Because there is no production without the firm, this sum is the firm's social marginal product, that is, its VCG transfer. To understand the connection between an agent's smallest Lindahl price and his VCG transfer, it is useful to view the VCG transfers as threshold payments. With that interpretation in mind, notice that at the top left and bottom right corners of the triangle, the utility of the other agent (William respectively Paul) is 0 and the firm's profit is 0. Thus, the smallest value an agent can report in the VCG mechanism without inducing the allocation to change – his VCG transfer – is indeed his smallest Lindahl price.

In the remainder of the paper, we show, among other results, that Lindahl prices exist for general social choice problems and that the equivalence between the agents' smallest Lindahl prices and the firm's largest sum of Lindahl prices and VCG transfers extends to these general problems.

3 Setup

We consider an economy with a firm, denoted by f, a finite set of agents $\mathcal{N}$, whose typical element we denote by i, with $N := |\mathcal{N}| \geq 1$, and a finite set of social alternatives $\mathcal{A}$, whose typical element we denote by a, with $A = |\mathcal{A}|$ satisfying $A \geq 2$. Producing alternative $a \in \mathcal{A}$ involves the cost $c^a \in [0, \overline{c}]$ with $\overline{c} > 0$ for the firm; let $\mathbf{c} = (c^a)_{a\in\mathcal{A}}$. Likewise, agent i's value for alternative a is denoted v_i^a and assumed to satisfy $v_i^a \in [0, \overline{v}]$ with $\overline{v} > 0$; let $\mathbf{v}_i = (v_i^a)_{a\in\mathcal{A}}$

and $\mathbf{v} = (\mathbf{v}_i)_{i \in \mathcal{N}}$. We assume that $\mathcal{A}$ contains the null alternative, or outside option, a_0 at which every agent and the firm obtain their status quo payoffs, which we normalize to 0; that is, we assume $c^{a_0} = 0$ and $v_i^{a_0} = 0$ for all $i \in \mathcal{N}$. The agents and the firm have quasilinear utility, so that if alternative a is chosen and the firm is paid t, the firm's payoff is $t - c^a$ while agent i's payoff when paying t and the alternative is a is $v_i^a - t$.

Given values and costs $(\mathbf{v}, \mathbf{c})$, the set of efficient allocations, denoted $\mathcal{A}^*$, contains all allocations that maximize the sum of the agents' values minus the firm's cost; that is,

$$\mathcal{A}^* = \arg\max_{a \in \mathcal{A}} \sum_{i \in \mathcal{N}} v_i^a - c^a.$$

As the set of alternatives is finite, the set of efficient alternatives is nonempty; however, it may be multi-valued. Letting $a^* \in \mathcal{A}^*$ be any efficient allocation, the maximized (*social*) *welfare*, denoted W, is the sum of values minus costs at an efficient allocation; that is,

$$W = \sum_{i \in \mathcal{N}} v_i^{a^*} - c^{a^*}.$$

Note that W does not depend on a^* since all efficient allocations yield the same welfare.

We assume that $\bar{c} \geq N\bar{v}$, which means that if the firm has sufficiently high costs for all alternatives (except the outside option), then it is efficient to provide the outside option alternative a_0 even if all the agents have the highest possible values for every alternative $a \in \mathcal{A} \setminus \{a_0\}$.

Suppose now that agent $i \in \mathcal{N}$ is not present (or equivalently, $\mathbf{v}_i = \mathbf{0}$). Then, the set of efficient alternatives becomes

$$\mathcal{A}^*_{-i} = \arg\max_{a \in \mathcal{A}} \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^a - c^a,$$

Letting $a^*_{-i} \in \mathcal{A}^*_{-i}$ be any efficient alternative when i is absent, the welfare in the economy in which agent i is absent is

$$W_{-i} = \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{a^*_{-i}} - c^{a^*_{-i}},$$

which also does not depend on which efficient alternative is picked, since all of them give by definition the same level of welfare.

Our model offers a wide range of interpretations and applications. For example, the alternatives $a \in \mathcal{A}$ could be public goods that differ with respect to scale and scope, in which case, generically, one would expect $v_i^a \neq v_i^{a'}$ for all $a, a' \in \mathcal{A}$ with $a \neq a'$. As a case in point, a could be building a public swimming pool and a' building a public swimming pool and a bridge.[3] On the other hand, the alternatives could be different feasible allocations of a given set of private goods, which by definition exhibit rivalry in consumption and permit exclusion. In contrast to public goods, we now have $v_i^a = v_i^{a'}$ for all a, a' that do not differ in the bundle of goods i obtains.[4]

3.1 Lindahl Price Vectors

In the classic environment underlying Lindahl prices, all agents and the firm are price takers and choose a social alternative that maximizes their payoff. As with Walrasian prices, where these prices come from is not part of the model. To fix ideas, they may be thought of as being set by a fictitious and benevolent auctioneer who has complete information about the agents' values and the firm's costs.[5]

A *price vector* specifies, for each agent, a non-negative price of each non-null alternative and a zero price for the null alternative; that is, the set of price vectors is $\{0\}^N \times \mathbb{R}_{\geq 0}^{N\cdot(A-1)}$. Given values and costs $(\mathbf{v}, \mathbf{c})$, a *Lindahl price vector* is a price vector that *supports* an efficient allocation $a^* \in \mathcal{A}$ in the sense that, under that price vector, (i) it is optimal for every agent $i \in \mathcal{N}$ to choose a^*, (i.e., choosing a^* maximizes i's payoff) and (ii) it is optimal for the firm to provide a^* (i.e., providing a^* maximizes the firm's payoff, or profit. Formally, a price vector $\boldsymbol{\lambda}$ is a Lindahl price vector if, for some $a^* \in \mathcal{A}$,

$$a^* \in \arg\max_{a \in \mathcal{A}} v_i^a - \lambda_i^a \text{ for every } i \in \mathcal{N} \quad \text{and} \tag{1}$$

$$a^* \in \arg\max_{a \in \mathcal{A}} \sum_{i \in \mathcal{N}} \lambda_i^a - c^a. \tag{2}$$

Note that $\lambda_i^{a^*}$ is the Lindahl price agent i pays, which we will refer to as agent i's *effective* Lindahl price.

[3] As both the swimming pool and the bridge admit, in principle, exclusion the model also encompasses "public goods with exclusion," which are sometimes also referred to as *club goods*.

[4] We formally define the private goods environment in Section 5.2

[5] This set of assumptions is, of course, subject to a criticism that is familiar from debates related to the Walrasian model. For example, Samuelson (1955, p. 354) writes that "there is something circular and unsatisfactory" about constructions like Lindahl's: "[T]hey show what the final equilibrium looks like, but by themselves they are not generally able to find the desired equilibrium."

3.2 VCG Mechanism

The informational assumptions underlying the VCG mechanism are fundamentally different from those used in the analysis of Lindahl prices. In particular, every agent $i \in \mathcal{N}$ is now assumed to be privately informed about his type $\mathbf{v}_i = (v_i^a)_{a\in\mathcal{A}}$ and the firm to be privately informed about its cost $\mathbf{c} = (c^a)_{a\in\mathcal{A}}$ except, of course, that $v_i^{a_0} = 0 = c^{a_0}$ is commonly known.

In lieu of an auctioneer who sets prices given knowledge about $\mathbf{v}$ and $\mathbf{c}$, the stipulation is now that there is a mechanism designer who uses the direct mechanism $\langle \alpha, \boldsymbol{\tau} \rangle$ that asks the agents to report their values and the firm its costs. The function $\alpha : [0, \overline{v}]^{NA} \times [0, \overline{c}]^A \to \mathcal{A}^*$ is the *social choice rule* that selects an efficient alternative $a^* \in \mathcal{A}^*$ and $\boldsymbol{\tau} : [0, \overline{v}]^{NA} \times [0, \overline{c}]^A \to \mathbb{R}^{N+1}$ is the *transfer rule* with, for $i \in \mathcal{N}$, $\underline{\tau}_i^{a^*}$ being the transfer from agent i to the designer and $\overline{\tau}_f^{a^*}$ being the transfer from the designer to the firm when the selected efficient alternative is $a^* \in \mathcal{A}^*$. (Though implicit in our notation, $\mathcal{A}^*$ and thus a^* are defined with respect to the reported values and costs $(\mathbf{v}, \mathbf{c})$.) The VCG transfer paid by agent $i \in \mathcal{N}$ is i's externality on other agents; that is, the welfare of others when i is not present minus the welfare of others when i is present:

$$\underline{\tau}_i^{a^*} = \underbrace{\left(\sum_{j\in\mathcal{N}\setminus\{i\}} v_j^{a^*_{-i}} - c^{a^*_{-i}} \right)}_{\text{Welfare of others when } i \text{ is not present, } W_{-i}} - \underbrace{\left(\sum_{j\in\mathcal{N}\setminus\{i\}} v_j^{a^*} - c^{a^*} \right)}_{\text{Welfare of others when } i \text{ is present, } W - v_i^{a^*}} \geq 0. \tag{3}$$

The inequality in (3) follows from the fact that a^*_{-i} is an allocation that is efficient when agent i is not present. Note that, since the VCG transfer paid by agent i can be written as $\underline{\tau}_i^{a^*} = W_{-i} - (W - v_i^{a^*})$, it depends on which efficient allocation is picked when i is present, but not on which efficient allocation is picked when i is absent. The payoff of agent i under the VCG mechanism is $v_i^{a^*} - \underline{\tau}_i^{a^*} = W - W_{-i}$ and does not depend on which efficient allocation is chosen when i is present.

The *VCG transfer* that the firm receives is equal to the externality of the firm on the agent; that is, the agents' welfare when f is present minus the agents' welfare when f is not present:

$$\overline{\tau}_f^{a^*} = \underbrace{\sum_{i\in\mathcal{N}} v_i^{a^*}}_{\text{Agents' welfare when } f \text{ is present}} - \underbrace{0}_{\text{Agents' welfare when } f \text{ is not present}} \geq 0. \tag{4}$$

Intuitively, the firm is paid for the value its presence creates for the agents. The welfare of the agents when the firm is absent (or equivalently, $\mathbf{c} = (\overline{\mathbf{c}})$) is zero since nothing is produced, and therefore only the null alternative is available. It is evident from (4) that the firm's VCG transfer at efficient allocation a^* is positive if and only if there exists an agent i such that $v_i^{a^*} > 0$. The firm's VCG transfer at a^* can be written as $\overline{\tau}_f^{a^*} = W + c^{a^*}$ and the firm's payoff under the VCG mechanism is $\overline{\tau}_f^{a^*} - c^{a^*} = W$.

As is well known, the VCG mechanism is the ex-post efficient, dominant strategy incentive compatible (DIC), ex-post individually rational (EIR) mechanism that minimizes the designer's deficit (i.e., the difference between the transfer to the firm and the sum of the transfers from the agents).[6] A mechanism is DIC if, for all $i \in \mathcal{N}$, all $\mathbf{v}_i, \hat{\mathbf{v}}_i \in [0, \overline{v}]^A$, all $\mathbf{v}_{-i} \in [0, \overline{v}]^{(N-1)A}$ and all $\mathbf{c} \in [0, \overline{c}]^A$:

$$v_i^{\alpha(\mathbf{v}_i, \mathbf{v}_{-i}, \mathbf{c})} - \underline{\tau}_i(\mathbf{v}_i, \mathbf{v}_{-i}, \mathbf{c}) \geq v_i^{\alpha(\hat{\mathbf{v}}_i, \mathbf{v}_{-i}, \mathbf{c})} - \underline{\tau}_i(\hat{\mathbf{v}}_i, \mathbf{v}_{-i}, \mathbf{c})$$

and, for all $\mathbf{c}, \hat{\mathbf{c}} \in [0, \overline{c}]^A$ and all $\mathbf{v} \in [0, \overline{v}]^{NA}$:

$$\overline{\tau}_f(\mathbf{c}, \mathbf{v}) - c^{\alpha(\mathbf{v}, \mathbf{c})} \geq \overline{\tau}_f(\hat{\mathbf{c}}, \mathbf{v}) - c^{\alpha(\mathbf{v}, \hat{\mathbf{c}})}.$$

That is, for any reported type profile of all others, every agent and the firm are better off reporting their true types than reporting anything else.

A mechanism satisfies EIR if for all possible type profiles the payoff of every agent $i \in \mathcal{N}$ and the firm f is non-negative; that is, it is at least as high as the payoff from the outside option, or null alternative, a_0. A mechanism satisfying DIC and EIR is ex-post efficient if it always chooses an efficient allocation, that is, if $\alpha(\mathbf{v}, \mathbf{c}) \in \mathcal{A}^*$ for all $(\mathbf{v}, \mathbf{c})$.

4 The Lindahl-VCG Equivalence

Observe that Lindahl price vectors satisfy complete-information analogues to the incentive compatibility and individual rationality constraints insofar as they give every agent and the

[6] Holmström (1979) showed that DIC and ex post efficiency imply that the mechanism has to be a Groves' mechanism (Groves, 1973), given a smoothly connected type space, which is an assumption that is satisfied in our setting. Making the EIR constraint hold with equality for the worst-off types then means that the mechanism maximizes revenue subject to efficiency, DIC and EIR. The direct mechanism that does this is the VCG mechanism.

firm the incentive to choose an efficient alternative a^* and make participation preferable to walking away with the outside option.

We next establish two properties of the set of Lindahl price vectors, which we denote by Λ. First, the set of Lindahl price vectors is independent of which efficient alternative is chosen. Therefore, it can equivalently be defined as the set of price vectors that supports *an* efficient allocation or the set of price vectors that supports *all* efficient allocations.[7] Second, the set of Lindahl price vectors is nonempty (and compact), which implies that an auctioneer who knows the values and cost for each alternative can always find prices that clear the market.

Proposition 1.

(a) Every Lindahl price vector supports every efficient allocation.

(b) The set of Lindahl price vectors is nonempty and compact.

Given any efficient allocation a^*, agent i's ($i \in \mathcal{N}$) *smallest Lindahl price* is the smallest price for a^* that i can face in any Lindahl price vector (i.e., the smallest effective Lindahl price):

$$\underline{\lambda}_i^{a^*} = \min_{\boldsymbol{\lambda} \in \Lambda} \lambda_i^{a^*}.$$

The firm's *Lindahl revenue* given $\boldsymbol{\lambda}$ is simply $\sum_{i \in \mathcal{N}} \lambda_i^{a^*}$. Accordingly, its *largest* Lindahl revenue, denoted $\overline{\lambda}_f^{a^*}$, is the largest revenue for providing the efficient alternative a^* that the firm can collect in any Lindahl price vector, that is,

$$\overline{\lambda}_f^{a^*} = \max_{\boldsymbol{\lambda} \in \Lambda} \sum_{i \in \mathcal{N}} \lambda_i^{a^*}.$$

As the set of Lindahl price vectors is a nonempty, compact set (by part (b) of Proposition 1), each agent's smallest Lindahl price and the firm's largest Lindahl revenue are well defined. It is worth noting that, while the set of Lindahl price vectors does not depend on which efficient alternative is chosen, the smallest Lindahl prices and the largest Lindahl revenue do; that is, for another efficient alternative $a^\sharp \in \mathcal{A}^* \setminus \{a^*\}$, it may be that $\underline{\lambda}_i^{a^\sharp} \neq \underline{\lambda}_i^{a^*}$ for some $i \in \mathcal{N}$ or $\overline{\lambda}_f^{a^\sharp} \neq \overline{\lambda}_f^{a^*}$.

[7]See also Corollary 1 in Bikhchandani and Mamer (1997) and Claim 1 in Delacrétaz et al. (2022) for a similar result in a private good setting.

The proof that the set of Lindahl price vectors is nonempty is constructive. By setting all prices as high as possible, i.e., $\lambda_i^a = v_i^a$ for all $i \in \mathcal{N}$ and all $a \in \mathcal{A}$, each agent's payoff is zero no matter what he chooses, so they all might as well choose the efficient allocation a^*. At this price vector, the firm's payoff at allocation a is $\sum_{i \in \mathcal{N}} v_i^a - c^a$; that is, welfare at allocation a. Thus, the firm's profit-maximization problem is the same as that of maximizing welfare. Hence, the efficient allocation a^* is profit-maximizing. Observe also that $\max_{a \in \mathcal{A}} \sum_{i \in \mathcal{N}} v_i^a - c^a$ is precisely the problem the firm faces in the VCG mechanism, which makes the firm the residual claimant to the social surplus it generates.[8] Thus, the firm's largest Lindahl revenue is its transfer in the VCG mechanism, that is, $\overline{\tau}_f^{a^*} = \overline{\lambda}_f^{a^*} = \sum_{i \in \mathcal{N}} v_i^{a^*}$.

Conversely, consider the relationship between Lindahl prices and VCG transfers for the agents. To develop intuition, assume for simplicity that $\mathcal{A}$ only contains two elements and that the non-null alternative is the efficient one. Let i's Lindahl price for a^* be $\underline{\lambda}_i^{a^*} = \max\{c^{a^*} - \sum_{j \neq i} v_j^{a^*}, 0\}$ and, as always, let $\lambda_i^{a_0} = 0$. Agent i's problem is then to choose $a \in \mathcal{A}$ to maximize $v_i^a - \lambda_i^a$. Temporarily neglecting the constraint imposed by the max, observe that

$$v_i^{a^*} - \left(c^{a^*} - \sum_{j \neq i} v_j^{a^*}\right) = \sum_{j \in \mathcal{N}} v_j^{a^*} - c^{a^*},$$

which is positive because $a^* \neq a_0$ is, by assumption, the efficient alternative. Hence, if $\underline{\lambda}_i^{a^*} = c^{a^*} - \sum_{j \neq i} v_j^{a^*}$, i will optimally choose the efficient alternative. If, on the other hand, $\underline{\lambda}_i^{a^*} = 0$, then i's payoff from choosing a^* is $v_i^{a^*}$, which by assumption is non-negative. Thus, a^* remains the optimal choice for i. Because $\max\{c^{a^*} - \sum_{j \neq i} v_j^{a^*}, 0\}$ is the VCG transfer for agent i in a binary public good problem, it follows that $\underline{\tau}_i^{a^*} = \underline{\lambda}_i^{a^*}$ for binary allocation problems.

When $\mathcal{A}$ contains more than two alternatives, the arguments for the agents become more involved because, in contrast to the firm, the efficient allocation without agent i is not necessarily a_0 while the default option a_0 is the only alternative available without the firm. Nevertheless, there is a simple intuition as to why the equivalence between the agents' small-

[8] We have formulated the VCG mechanism as a mechanism that asks the participants to report their types and then chooses an alternative and transfers. Alternatively and equivalently, one can formulate the VCG mechanism as two-stage mechanism that first asks the participants to report their types and then offers each of them a menu of transfers, one for each alternative $a \in \mathcal{A}$, and lets each of them choose an alternative. It is in this formulation of the VCG mechanism that the firm faces the problem of $\max_{a \in \mathcal{A}} \sum_{i \in \mathcal{N}} v_i^a - c^a$.

est Lindahl prices and their VCG transfers extends. A Lindahl price vector that minimizes the price i pays for a^* must be such that at allocation a^* the agents other than i and the firm jointly receive a surplus of W_{-i}, the largest surplus they could achieve without i. If they receive less, at least one of those agents or the firm would prefer a^*_{-i} over a^*. If they receive more, then it is possible to lower i's price for a^* while continuing to make sure that the other agents and the firm prefer a^* over all other alternatives. Therefore, i's surplus at a^* is $W - W_{-i}$, which is the payoff i obtains in the VCG mechanism; hence, the price i pays equals his VCG transfer.[9] We next formalize this equivalence.

Theorem 1. *For any efficient alternative $a^* \in \mathcal{A}^*$:*

(a) *The smallest Lindahl price of each agent i is equal to agent i's VCG transfer, $\underline{\lambda}_i^{a^*} = \underline{\tau}_i^{a^*}$ for every $i \in \mathcal{N}$;*

(b) *The largest Lindahl revenue of the firm is equal to the firm's VCG transfer, $\overline{\lambda}_f^{a^*} = \overline{\tau}_f^{a^*}$.*

A key insight from Theorem 1 is that incentivizing (making it a dominant strategy for) any agent to reveal their private information is possible by selecting the most favorable Lindahl price vector for this agent. However, as we shall see, it follows from Proposition 3 that providing these incentives for all agents simultaneously is only possible if there is a unique Lindahl price vector.

Theorem 1 also sheds new light on the deficit that the market maker "typically" incurs in the VCG mechanism. The deficit arises because the prices that participants face come from different Lindahl price vectors; the prices paid by the agents must be as low as possible, and the prices collected by the firm must be as high as possible. We elaborate on this insight in Section 6.

Recall that our model also encompasses as a special case the problem of allocating private goods by specifying that each alternative represents an allocation in which each agent receives a bundle of private goods and the firm produces all those bundles. Theorem 1 therefore applies to that environment: Each agent's smallest Lindahl price equals the VCG transfer he pays and the firm's largest Lindahl revenue equals the VCG transfer it receives.

[9] That is, $v_i^{a^*} - \underline{\lambda}_i^{a^*} = W - W_{-i}$ so $\underline{\lambda}_i^{a^*} = W_{-i} - W + v_i^{a^*} = \underline{\tau}_i^{a^*}$.

This contrasts with results on Walrasian prices and VCG transfers. For example, the analyses in Gul and Stacchetti (1999) and Delacrétaz et al. (2022) imply that, in general, the set of Walrasian price vectors may be empty and, if it is nonempty, each agent's smallest Walrasian price is weakly larger than the VCG transfer he pays and the firm's largest Walrasian revenue is weakly smaller than the VCG transfer it receives. The discrepancy stems from a fundamental difference between Lindahl and Walrasian prices: Lindahl prices allow freely setting the price of each alternative for each agent, and indeed that flexibility is essential to Theorem 1. Walrasian prices, on the other hand, satisfy *anonymity* – any two agents who receive the same bundle pay the same price for that bundle – and *linearity* – the price that an agent pays for a bundle equals the sum of the prices of all elements in that bundle. Walrasian prices are therefore a *restricted* version of Lindahl prices, which drives the difference in how they relate to VCG transfers. In the next section, we formalize the impact on the relationship between VCG transfers and Lindahl prices of imposing restrictions, like anonymity and linearity, on the set of permissible Lindahl prices.

5 Price Restrictions

Recall that, for each agent, a price vector specifies the non-negative price of each non-null alternative and a zero price for the null alternative; the set of price vectors is thus $\{0\}^N \times \mathbb{R}_{\geq 0}^{N \cdot (A-1)}$. In many environments, however, additional restrictions are imposed on price vectors. For example, in an exchange economy, prices often must be anonymous and linear.

For that purpose, we now add a new ingredient to our model: a set of permitted or admissible price vectors $\mathcal{P} \subseteq \{0\}^N \times \mathbb{R}_{\geq 0}^{N \cdot (A-1)}$. The elements of $\mathcal{P}$ that satisfy conditions (1) and (2) form the set of *restricted Lindahl price vectors*, which we denote by $\Lambda(\mathcal{P})$. We first state the general effects of restrictions on the Lindahl price vectors and then specialize to the setting with private goods and Walrasian prices.

5.1 General Impact of Price Restrictions

Absent any restrictions, the set of Lindahl price vectors Λ is nonempty by Proposition 1. This may not be the case for the set of restricted Lindahl price vectors, since $\Lambda(\mathcal{P}) = \Lambda \cap \mathcal{P}$

and $\mathcal{P}$ may not contain any elements of Λ.[10] If $\Lambda(\mathcal{P})$ is nonempty and compact,[11] we define each agent i's *smallest restricted Lindahl price* for any efficient allocation $a^* \in \mathcal{A}^*$ to be the smallest price that i can face in any restricted Lindahl price vector:

$$\underline{\lambda}_i^{a^*}(\mathcal{P}) = \min_{\boldsymbol{\lambda} \in \Lambda(\mathcal{P})} \lambda_i^{a^*}.$$

We similarly define the firm's *largest restricted Lindahl revenue* to be the largest revenue that the firm can collect in any restricted Lindahl price vector:

$$\overline{\lambda}_f^{a^*}(\mathcal{P}) = \max_{\boldsymbol{\lambda} \in \Lambda(\mathcal{P})} \sum_{i \in \mathcal{N}} \lambda_i^{a^*}.$$

Theorem 2. *Suppose that the restricted set of Lindahl price vectors is nonempty, i.e., $\Lambda(\mathcal{P}) \neq \varnothing$, and compact. Then, for any efficient alternative $a^* \in \mathcal{A}^*$:*

(a) *The smallest restricted Lindahl price of each agent i is at least i's VCG transfer, $\underline{\lambda}_i^{a^*}(\mathcal{P}) \geq \underline{\tau}_i^{a^*}$ for every $i \in \mathcal{N}$; $\underline{\lambda}_i^{a^*}(\mathcal{P}) = \underline{\tau}_i^{a^*}$ if and only if $(\arg\min_{\boldsymbol{\lambda} \in \Lambda} \lambda_i^{a^*}) \cap \mathcal{P} \neq \varnothing$.*

(b) *The largest restricted Lindahl revenue of the firm is at most the firm's VCG transfer, $\overline{\lambda}_f^{a^*}(\mathcal{P}) \leq \overline{\tau}_f^{a^*}$; $\overline{\lambda}_f^{a^*}(\mathcal{P}) = \overline{\tau}_f^{a^*}$ if and only if $(\arg\max_{\boldsymbol{\lambda} \in \Lambda} \sum_{i \in \mathcal{N}} \lambda_i^{a^*}) \cap \mathcal{P} \neq \varnothing$.*

Theorem 2 provides the important insight that, as long as the set of Lindahl price vectors remains nonempty and compact, a relationship remains between each agent's smallest Lindahl price and their VCG transfer in the form of an inequality; the same holds for the firm's largest Lindahl revenue and its VCG transfer. The reason is that the set of Lindahl price vectors becomes smaller as restrictions are imposed on it. The key as to whether or not an agent's smallest Lindahl price (or the firm's largest Lindahl revenue) changes as a result of a price restriction is whether or not the price restriction takes away the agent's (or the firm's) most favorable Lindahl price vectors.

5.2 Price Restrictions with Private Goods

We now formalize how, in the private good setting, the set of Walrasian price vectors obtains from the set of Lindahl price vectors by keeping only anonymous and linear price vectors.

[10] To see that $\Lambda(\mathcal{P}) = \Lambda \cap \mathcal{P}$, note that whether or not a price vector satisfies (1) and (2) does not depend on $\mathcal{P}$. Therefore, the set of restricted Lindahl price vectors consist of those price vectors that are allowed and satisfy (1) and (2).

[11] $\Lambda(\mathcal{P})$ may not be compact if $\mathcal{P}$ is not compact.

In the private goods setting, there is a set of objects $\mathcal{O}$ with typical element o and cardinality $O = |\mathcal{O}|$. Each alternative a represents a partition $((B_i^a)_{i\in\mathcal{N}}, B_f^a)$ of the set $\mathcal{O}$, where B_i^a is the bundle of objects that each agent i receives at alternative a and B_f^a is the bundle of objects that is not allocated to any agents (hence not produced by the firm). Each possible partition of $\mathcal{O}$ is represented by an alternative; hence, there are as many alternatives as there are partitions of $\mathcal{O}$. The null alternative represents the partition in which no object is produced: $B_i^{a_0} = \varnothing$ for every $i \in \mathcal{N}$ and $B_f^{a_0} = \mathcal{O}$.

An *object price vector* $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}} \in \mathbb{R}^O_{\geq 0}$ specifies a price for each object. From any object price vector $\boldsymbol{p}$, it is possible to infer an (agent-alternative) price vector $\boldsymbol{\mu}(\boldsymbol{p}) \in \{0\}^N \times \mathbb{R}^{N\cdot(A-1)}_{\geq 0}$ by setting, for each $i \in \mathcal{N}$ and each $a \in \mathcal{A}$, $\mu_i^a(\boldsymbol{p}) = \sum_{o\in B_i^a} p_o$. The set of (agent-alternative) price vectors that can be inferred from an object price vector is

$$\mathcal{W} = \{\boldsymbol{\mu} \in \{0\}^N \times \mathbb{R}^{N\cdot(A-1)}_{\geq 0} : \boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{p}) \text{ for some } \boldsymbol{p} \in \mathbb{R}^O_{\geq 0}\}.$$

The set of *Walrasian price vectors* is the set of Lindahl price vectors that satisfy this restriction: $\Lambda(\mathcal{W}) = \Lambda \cap \mathcal{W}$.[12] Understanding that Walrasian prices, in the private goods setting, constitute a restriction of Lindahl prices establishes by Theorem 2 that if the set of Walrasian prices is nonempty, then each agent's smallest Walrasian price is weakly larger than the VCG transfer he pays and the firm's largest Walrasian revenue is weakly smaller than the VCG transfer it receives.

We next verify that the restrictions imposed by Walrasian prices are precisely anonymity and linearity. Anonymity requires that all agents pay the same price for the same bundle. Formally, a price vector $\boldsymbol{\mu} \in \{0\}^N \times \mathbb{R}^{N\cdot(A-1)}_{\geq 0}$ is *anonymous* if, for any $i, j \in \mathcal{N}$ and any $a, a' \in \mathcal{A}$ such that $B_i^a = B_j^{a'}$, we have $\mu_i^a = \mu_j^{a'}$. Linearity requires that the price that an agent pays for a bundle be the sum of what he pays for the individual objects in that bundle. In order to formally define linearity, it is useful, for each $i \in \mathcal{N}$ and any $o \in \mathcal{O}$, to denote by a_i^o the alternative in which i is allocated o (and no other objects) and no other agent is allocated anything; that is, $B_i^{a_i^o} = \{o\}$ and $B_j^{a_j^o} = \varnothing$ for every $j \in \mathcal{N} \setminus \{i\}$. A price vector $\boldsymbol{\mu} \in \{0\}^N \times \mathbb{R}^{N\cdot(A-1)}_{\geq 0}$ is *linear* if, for any $i \in \mathcal{N}$ and any $a \in \mathcal{A}$, $\mu_i^a = \sum_{o\in B_i^a} \mu_i^{a_i^o}$.

[12] Walrasian prices are typically defined as object price vectors. We use here the (agent-alternative) price vectors that can be inferred from them to facilitate the comparison with Lindahl prices. Note that $\mathcal{W}$ and hence $\Lambda(\mathcal{W})$ are compact sets.

Proposition 2. *For any price vector* $\boldsymbol{\mu} \in \{0\}^N \times \mathbb{R}_{\geq 0}^{N\cdot(A-1)}$, $\boldsymbol{\mu} \in \mathcal{W}$ *if and only if* $\boldsymbol{\mu}$ *is anonymous and linear.*

Since the set of Walrasian price vectors $\Lambda(\mathcal{W})$ is equal to $\Lambda \cap \mathcal{W}$, a direct consequence of Proposition 2 is that the Walrasian price vectors are the Lindahl price vectors that are anonymous and linear. Because of those restrictions, each participant's most favorable Walrasian price may be strictly less favorable than their VCG transfer.

Two special cases where Theorem 1 extends to Walrasian prices are when all agents have single-unit demand and the firm has single-unit supply, and when all agents have additive values and the firm has additive costs. In the former case, linearity has no bite because each agent receives at most one unit and anonymity has no bite because at most one agent receives a unit. With additive values and costs, efficiency dictates that either nothing be produced or that all produced units be assigned to the agent with the largest value; hence, the problem is equivalent to one with single-unit demand and supply. Theorem 1 also "partially" extends to Walrasian prices in the case of a homogeneous good market in which each agent has single-unit demand and the firm has weakly increasing marginal costs. It extends for the agents, as each agent's smallest Walrasian price is equal to the agent's VCG transfer. The VCG transfer of each agent who is assigned a unit equals either the highest value of the agents who are not assigned a unit – as first shown for environments without production costs by Demange (1982), Leonard (1983), and Gul and Stacchetti (1999) – or the firm's marginal cost of producing the last unit, whichever is larger.[13] Therefore, all agents who buy a unit have the same VCG transfer and anonymity has no bite.[14] As linearity has no bite either in this single-unit demand environment, Theorem 1 holds for the agents. However, it does not hold for the firm. As nothing is produced without the firm, its VCG transfer is the sum of the values of the agents who are assigned a unit. Anonymity restricts the Walrasian price to be the same for all agents; hence, the firm can only charge the smallest value of the agents who purchase a unit. Unless all of them have the same value, the firm's VCG transfer is

[13] To see this, note that the agent's externality on others is that his presence either prevents the next highest-value agent from getting a unit or forces the firm to produce one more unit.

[14] As Example 1 shows, that result can break down when the firm has decreasing marginal cost because removing an agent may reduce production by more than one unit, and therefore the agents may no longer have the same VCG transfers while anonymity continues to require they face the same smallest Walrasian price.

therefore larger than its largest Walrasian revenue.

We end this section with two examples, which show that either one of those restrictions can on its own cause this strict inequality. The first example shows that an agent's smallest anonymous Lindahl price may be larger than his VCG transfer and the firm's largest anonymous Lindahl revenue may be smaller than its VCG transfer.

Example 1. *There are two agents 1 and 2 and two (identical) objects o_1 and o_2. Agent 1 gets a value of 8 for either object and agent 2 gets a value of 6 from either object. Both agents have unit demand (i.e., adding the other object does not increase the value). The firm incurs a cost of 10 for producing either object and a cost of 11 for producing both.*

In Example 1, there are two efficient allocations: a^* assigns o_1 to agent 1 and o_2 to agent 2 ($B_1^{a^*} = \{o_1\}$ and $B_2^{a^*} = \{o_2\}$) while $a^\sharp$ assigns o_1 to agent 2 and o_2 to agent 1 ($B_1^{a^*} = \{o_2\}$ and $B_2^{a^*} = \{o_1\}$). The efficient level of welfare is $8 + 6 - 11 = 3$. If either agent is removed, it is no longer efficient to produce anything so the efficient level of welfare in that case is 0. The VCG transfers are

$$\underline{\tau}_1^{a^*} = \underline{\tau}_1^{a^\sharp} = 0 - (3 - 8) = 5, \quad \underline{\tau}_2^{a^*} = \underline{\tau}_2^{a^\sharp} = 0 - (3 - 6) = 3, \quad \overline{\tau}_f^{a^*} = \overline{\tau}_f^{a^\sharp} = 8 + 6 = 14.$$

Let $\boldsymbol{\lambda}$ be an anonymous Lindahl price vector. By anonymity, $\lambda_1^{a^*} = \lambda_2^{a^\sharp}$ (as $B_1^{a^*} = B_2^{a^\sharp} = \{o_1\}$) and $\lambda_1^{a^\sharp} = \lambda_2^{a^*}$ (as $B_1^{a^\sharp} = B_2^{a^*} = \{o_2\}$). By Proposition 1, $\boldsymbol{\lambda}$ supports both a^* and $a^\sharp$ so each agent must be indifferent between the two alternatives: $v_i^{a^*} - \lambda_i^{a^*} = v_i^{a^\sharp} - \lambda_i^{a^\sharp}$ for $i = 1, 2$. As $v_i^{a^*} = v_i^{a^\sharp}$, it follows that $\lambda_i^{a^*} = \lambda_i^{a^\sharp}$ for $i = 1, 2$. Then, the price of both alternatives must be the same for both agents. Let $p = \lambda_1^{a^*} = \lambda_1^{a^\sharp} = \lambda_2^{a^*} = \lambda_2^{a^\sharp}$. As p must be small enough for agent 2 to buy an object but large enough for the firm to produce both objects, we have that $5.5 \leq p \leq 6$. Each agent's smallest anonymous Lindahl price is then obtained by setting p to its lower bound, hence:[15]

$$\underline{\lambda}_i^{a^*} = \underline{\lambda}_i^{a^\sharp} = 5.5 > 5 \geq \underline{\tau}_i^{a^*} = \underline{\tau}_i^{a^\sharp} \quad \text{for } i = 1, 2.$$

The firm's largest anonymous Lindahl revenue is obtained by setting p to its upper bound, hence

$$\overline{\lambda}_f^{a^*} = \overline{\lambda}_f^{a^\sharp} = 2 \cdot 6 = 12 < 14 = \overline{\tau}_f^{a^*} = \overline{\tau}_f^{a^\sharp}.$$

[15] For notational simplicity, we keep the anonymity restriction implicit, denoting agent i's smallest *anonymous* Lindahl price at allocation a^* by $\underline{\lambda}_i^{a^*}$.

Therefore, anonymity alone can make every participant's most favorable Lindahl price vector less favorable than the VCG transfer.[16]

We now turn to linearity and show that an agent's smallest linear Lindahl price may be larger than his VCG transfer and the firm's largest linear Lindahl revenue may be smaller than its VCG transfer.

Example 2. *There is one agent i and two (identical) objects o_1 and o_2. The agent's value for consuming either object is 4 and his value for consuming both is 7. The firm's cost for producing either object is 1 and its cost for producing both objects is 3.*

In Example 2, the (unique) efficient allocation a^* is the one in which both objects are produced ($B_i^{a^*} = \{o_1, o_2\}$) and the efficient level of welfare is 4 ($= 7 - 3$). If the agent or the firm is removed, nothing is produced and the efficient level of welfare in that case is 0. The VCG transfers are

$$\underline{\tau}_i^{a^*} = 0 - (4 - 7) = 3 \quad \text{and} \quad \overline{\tau}_f^{a^*} = 7.$$

Let $\boldsymbol{\lambda}$ be a linear Lindahl price vector and, for each $k = 1, 2$, denote by a_k the alternative where only object o_k is produced ($B_i^{a_k} = \{o_k\}$). Linearity dictates that $\lambda_i^{a^*} = \lambda_i^{a_1} + \lambda_i^{a_2}$. As $\boldsymbol{\lambda}$ must incentivize the agent to purchase both objects instead of just one of them, we have that

$$7 - \lambda_i^{a_1} - \lambda_i^{a_2} = 7 - \lambda_i^{a^*} \geq 4 - \lambda_i^{a_k} \quad \text{for every } i = 1, 2,$$

which implies that $\lambda_i^{a_1}, \lambda_i^{a_2} \leq 3$. It follows that

$$\lambda_i^{a^*} = \lambda_i^{a_1} + \lambda_i^{a_2} \leq 6 < 7 = \overline{\tau}_f^{a^*};$$

therefore, the firm's largest linear Lindahl revenue is smaller than its VCG transfer. As $\boldsymbol{\lambda}$ must incentivize the firm to produce both objects instead of just one of them, we have that

$$\lambda_i^{a_1} + \lambda_i^{a_2} - 3 = \lambda_i^{a^*} - 3 \geq \lambda_i^{a_k} - 1 \quad \text{for every } i = 1, 2,$$

[16] To construct an example in which the set of anonymous Lindahl prices is empty, simply subtract 2 from each value and cost. The price for each agent and each of the two efficient alternatives is again the same; letting it be p, we need $p \leq 4$ to incentivize agent 2 to buy an object and $p \geq 4.5$ to incentivize the firm to produce both objects.

which implies that $\lambda_i^{a_1}, \lambda_i^{a_2} \geq 2$. It follows that

$$\lambda_i^{a^*} = \lambda_i^{a_1} + \lambda_i^{a_2} \geq 4 > 3 = \underline{\tau}_i^{a^*};$$

therefore, the agent's smallest linear Lindahl price is larger than his VCG transfer.[17]

An application where linearity may be imposed on its own is the provision of a public good in multiple discrete units. One may set a price per unit and require that the price of each alternative be equal to the number of units produced multiplied by the unit price. Example 2 shows that Theorem 1 fails under that restriction. The general reason for this failure is that the VCG transfer of the firm must be equal to the total agents' welfare; as if the firm engaged in first-degree price discrimination, charging each agent the marginal value of each unit produced. To induce each agent to demand the efficient quantity, the highest unit price that each agent can be charged is his marginal value for the last unit produced. Hence, if the Lindahl price paid by the agents must be linear, then the maximum revenue of the firm is the sum over each agent of the product of the units produced and the agent's marginal value for the last unit. This is less than the firm's VCG transfer, as long as the marginal values of the units of the public good are decreasing (strictly for at least one agent for at least one produced unit) and so Theorem 1 fails but, as shown by Theorem 2, the relationship between extremal, linear Lindahl prices and VCG transfers persists as an inequality.

6 VCG Deficit and Cost Recovery

Two classic questions are of considerable theoretical and practical interest: When does implementing an efficient allocation require running a deficit if both the firm and the agents must be incentivized to reveal their private information and to participate? Relatedly, if only the agents have private information (e.g., because the firm is the designer, say, a social planner interested in efficiency), when can the production costs be recovered? In the next two subsections, we use properties of Lindahl price vectors to shed new light on these questions.

[17] To construct an example in which the set of linear Lindahl price vectors is empty, set the agent's values to 4 for consuming either object and 6 for consuming both, and set the firm's costs to 4 for producing either object and 5 for producing both. Incentivizing the agent to purchase both objects rather than one requires setting the price of each object to at most 2 ($= 6 - 4$) but incentivizing the firm to produce both objects rather than none requires setting the price of each object to at least 2.5 ($= 5/2$).

Recall that, in general, $\boldsymbol{\lambda}_i$ is an A-dimensional vector. Because for a fixed efficient allocation a^*, any agent i only pays the scalar $\lambda_i^{a^*}$. What matters for cost recovery and the deficit under the VCG mechanism are the effective Lindahl prices $\lambda_i^{a^*}$ introduced at the end of Section 3.1. Formally, for any efficient alternative $a^* \in \mathcal{A}^*$, we let $L^{a^*} = \{\boldsymbol{\ell} \in \mathbb{R}^N_{\geq 0} : \boldsymbol{\ell} = (\lambda_i^{a^*})_{i \in \mathcal{N}}$ for some $\boldsymbol{\lambda} \in \Lambda\}$ be the set of *effective Lindahl price vectors* under alternative a^*. An effective Lindahl price vector is constructed from any Lindahl price vector by only taking, for each agent, the price that the agent pays for the efficient alternative, i.e., the price that the agent effectively pays under that price vector. Each Lindahl price vector generates an effective Lindahl price vector but distinct Lindahl price vectors generate the same effective price vector if their entries associated with the efficient alternative a^* are the same.

6.1 No VCG deficit

The deficit of the VCG mechanism is:

$$D^{VCG} = \overline{\tau}_f^{a^*} - \sum_{i \in \mathcal{N}} \underline{\tau}_i^{a^*}.$$

Note that, by (3) and (4),

$$D^{VCG} = \sum_{i \in \mathcal{N}} v_i^{a^*} - \sum_{i \in \mathcal{N}} W_{-i} + \sum_{i \in \mathcal{N}} \left(W - v_i^{a^*}\right) = \sum_{i \in \mathcal{N}} (W - W_{-i}) \geq 0. \quad (5)$$

Therefore, the VCG deficit is nonnegative and does not depend on which efficient allocation is selected. We show that whether or not the VCG mechanism runs a (non-zero) deficit depends on whether or not the set of effective Lindahl price vectors is multi-valued.

Proposition 3. *For every efficient alternative $a^* \in \mathcal{A}^*$, $D^{VCG} = 0$ if and only if $|L^{a^*}| = 1$.*

Recall that the set of Lindahl price vectors Λ is nonempty, which directly implies that, for any $a^* \in \mathcal{A}$, L^{a^*} is nonempty. Since the VCG deficit is nonnegative (equation (5)), it follows that Proposition 3 can be equivalently formulated as $D^{VCG} > 0$ if and only if $|L^{a^*}| > 1$. Moreover, as the VCG deficit does not depend on the efficient alternative picked, a direct implication of Proposition 3 is that, for any two efficient alternatives $a^*, a^\sharp \in \mathcal{A}^*$, $|L^{a^*}| = 1$ if and only if $|L^{a^\sharp}| = 1$ (and $|L^{a^*}| > 1$ if and only if $|L^{a^\sharp}| > 1$). Therefore, even though the set of effective Lindahl prices depends on which efficient allocation is picked, the relationship between VCG deficit and effective Lindahl prices outlined in Proposition 3 does not.[18]

Proposition 3 connects the property of whether L^{a^*} is multi- or single-valued to whether or not the VCG mechanism runs a deficit. As such, it relates to the result, in a different setting, of Segal and Whinston (2016) that the *marginal core*[19] being multi-valued is sufficient for the VCG mechanism to run a deficit. In Appendix B we show that, in our setting, the marginal core being multi-valued is also necessary for the VCG mechanism to run a deficit. This is because each effective Lindahl price vector generates payoffs that are in the marginal core and the set of effective Lindahl price vectors contains a unique element if and only if the marginal core does (Proposition 5 in Appendix B).

If there is a unique Lindahl price vector, then there is also, by definition, a unique effective Lindahl price vector. Therefore, a direct consequence of Proposition 3 is that if there is a unique Lindahl price vector, then the VCG deficit is zero. As the next example shows, the converse does not hold: The set of Lindahl price vectors may be multi-valued when the set of effective Lindahl price vectors is not; hence, the VCG deficit may be zero even though the set of Lindahl price vectors is multi-valued.

Example 3. *There are one agent and three alternatives. The values and costs are*

$$\begin{array}{cc} & \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \begin{array}{c} v \\ c \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

In Example 3, there are two efficient alternatives: a_0 and a_1. The set of Lindahl price vectors contains all vectors $(\lambda^{a_0}, \lambda^{a_1}, \lambda^{a_2}) = (0, 1, x)$ with $x \in [0, 1]$; hence, the set of Lindahl price vectors is multi-valued. However, the set of effective Lindahl price vectors corresponding to either efficient allocation contains a unique element: $L^{a_0} = \{0\}$ and $L^{a_1} = \{1\}$. As Proposition 3 predicts, the VCG deficit is 0 since $\underline{\tau}^{a_0} = \overline{\tau}^{a_0} = 0$ and $\underline{\tau}^{a_1} = \overline{\tau}^{a_1} = 1$.

[18]Note that, since the price of the null alternative a_0 is zero for all agents, whenever a_0 is efficient there can only be one effective Lindahl price vector and hence the VCG mechanism does not run a deficit.

[19]They defined the marginal core as the set of payoffs that ensures no coalition of all but one participant gains from deviating.

6.2 Cost Recovery under VCG

If the revenue that can be extracted from the agents under the VCG mechanism does not cover the firm's cost, then allocating efficiently generates a monetary loss even if the firm does not have to be incentivized to reveal its private information (e.g., because the planner runs the firm). We say that the VCG mechanism achieves *cost recovery* at an efficient allocation $a^* \in \mathcal{A}$ if $\sum_{i \in \mathcal{N}} \underline{\tau}_i^{a^*} \geq c^{a^*}$. Note that

$$\begin{aligned} \sum_{i \in \mathcal{N}} \underline{\tau}_i^{a^*} - c^{a^*} &= \sum_{i \in \mathcal{N}} \big(W_{-i} - (W - v_i^{a^*}) \big) - c^{a^*} \\ &= \sum_{i \in \mathcal{N}} W_{-i} - NW + \sum_{i \in \mathcal{N}} v_i^{a^*} - c^{a^*} \\ &= \sum_{i \in \mathcal{N}} W_{-i} - (N-1)W, \end{aligned} \tag{6}$$

which does not depend on a^*. Therefore, whether or not the VCG mechanism achieves cost recovery is independent of which efficient allocation is picked.

For each efficient allocation $a^* \in \mathcal{A}^*$, define the set $\Lambda_C^{a^*}$ of *cost-recovery Lindahl – or C-Lindahl – price vectors* as the set containing all price vectors that satisfy condition (1) and, instead of (2),

$$\sum_{i \in \mathcal{N}} \lambda_i^{a^*} - c^{a^*} \geq 0. \tag{7}$$

We then define the set of *effective C-Lindahl price vectors* to be $L_C^{a^*} = \{ \boldsymbol{\ell} \in \mathbb{R}^N_{\geq 0} : \boldsymbol{\ell} = (\lambda_i^{a^*})_{i \in \mathcal{N}} \text{ for some } \boldsymbol{\lambda} \in \Lambda_C^{a^*} \}$. That is, each effective C-Lindahl price vector contains the price that each agent pays for a^* in some C-Lindahl price vector. Finally, we denote by $\underline{\boldsymbol{\lambda}}^{a^*} = (\underline{\lambda}_i^{a^*})_{i \in \mathcal{N}}$ the vector of smallest Lindahl prices. We next show that these concepts are key to cost recovery as the VCG mechanism achieves cost recovery if and only if the vector of smallest Lindahl prices is an effective C-Lindahl price vector.

Proposition 4. *For every efficient allocation $a^* \in \mathcal{A}^*$, $\underline{\boldsymbol{\lambda}}^{a^*} \in L_C^{a^*}$ if and only if $\sum_{i \in \mathcal{N}} \underline{\tau}_i^{a^*} \geq c^{a^*}$.*

While each of the set of effective C-Lindahl price vectors, the VCG transfers, and the cost of producing depends on which efficient allocation is picked, equation (6) ensures that Proposition 4 does not. That is, either the vector of smallest Lindahl prices belongs to the set of effective C-Lindahl price vectors and the VCG mechanism achieves cost recovery for all efficient allocations, or the vector of smallest Lindahl prices does not belong to the set of effective C-Lindahl price vectors and the VCG mechanism does not achieve cost recovery for any efficient allocation.

Note that, for every $a^* \in \mathcal{A}^*$, $L^{a^*} \subseteq L_C^{a^*}$ because condition (7) is less demanding than condition (2). It follows from Proposition 4 that $\underline{\boldsymbol{\lambda}}^{a^*} \in L^{a^*}$ is a sufficient condition for cost recovery. Moreover, $\underline{\boldsymbol{\lambda}}^{a^*} \in L^{a^*}$ is also necessary when there are only two alternatives (one of which is the null) because in that case, $L^{a^*} = L_C^{a^*}$.[20]

Our example from Section 2 illustrates Proposition 4. Recall that the VCG transfers are 2 for Paul and 1 for William but the cost to Erik is 4, hence the VCG mechanism does not achieve cost recovery. The set of effective C-Lindahl price vectors is represented in Figure 1 by the shaded triangle and does not contain the vector (2,1) of smallest Lindahl prices. A key intuition as to why this is the case lies in the free-riding problem that is inherent to a public good: The more one agent pays, the less the other has to pay to recover the cost, and therefore the cost is not recovered if each agent pays his smallest possible Lindahl price. It is easy to see that the result generalizes to any single-unit public good setting in which production is efficient because each agent's smallest Lindahl price occurs when the other agents are paying as much of the production cost as their values allow.

The free-riding problem disappears with private goods. Consider an economy with a homogeneous private good in which the agents have single-unit demands and the firm's marginal cost is nondecreasing. Each agent who is efficiently allocated a unit must pay at least the marginal cost of the last unit produced. The vector of smallest Lindahl prices (and by Theorem 1 the vector of VCG transfers) therefore assigns to each agent who is allocated a unit the marginal cost of the last unit produced. That vector is an effective C-Lindahl price vector because, at that price, (i) every agent who is efficiently allocated a unit is incentivized to purchase one, (ii) every agent who is efficiently not allocated a unit is incentivized not to purchase one, and (iii) the firm is incentivized to produce the number of units required to serve all agents who are efficiently allocated a unit. As Proposition 4 predicts, the VCG mechanism recovers the cost of production since nondecreasing marginal costs guarantee the

[20] If the null alternative is efficient, then $L^{a^*} = L_C^{a^*} = \{\mathbf{0}\}$ and if the non-null alternative is not efficient, then conditions (2) and (7) are identical.

sum of the VCG transfers is greater than or equal to the total cost of producing.

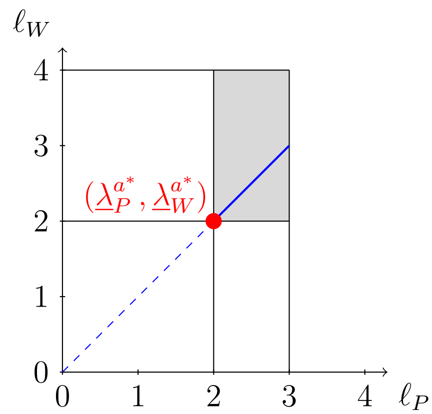

Figure 2: The set of effective C-Lindahl prices (shaded area), the vector of smallest Lindahl prices (red dot) and the set Walrasian price vectors (solid blue line).

As an illustration, consider a private good variant of the example in Section 2, in which Erik's marginal cost of producing the first and second copies of the book are $c \leq 2$ and 2, respectively (i.e., producing one copy costs c and producing two copies costs $c + 2$). Paul's valuation for a copy is 3 and William's is 4. Since each agent's valuation exceeds Erik's (i.e., the firm's) marginal cost of production for the second copy, the efficient allocation a^* is to allocate a unit of the book to each agent.[21] An effective C-Lindahl price vector must incentivize both agents to purchase a unit and the firm to produce the second unit. Therefore, the set of effective C-Lindahl price vectors contains all price vectors such that each agent's price is (i) larger than or equal to the marginal cost of producing the second unit (which is 2) and (ii) smaller than or equal to the agent's value (3 for Paul and 4 for William). The set is depicted in Figure 2.[22] Geometrically, the set of effective C-Lindahl price vectors in Figure 2 forms a rectangle whose bottom-left corner is the vector of smallest Lindahl prices. In contrast, the set of effective C-Lindahl price vectors in Figure 1 forms a triangle that does not contain a bottom-left corner. As our results show, that geometric difference drives the VCG mechanism's ability to recover the cost of production in one case

[21] Apart from the null and the efficient allocation, there are two allocations: produce the good for Paul only, a_1, and produce the good for William only, a_2. We have $c^{a_1} = c^{a_2} = c$, $v_P^{a_1} = v_P^{a^*} = 3$ and $v_W^{a_2} = v_W^{a^*} = 4$.

[22] Linearity does not restrict prices since agents have unit demand, but anonymity dictates that both agents face the same price. Therefore, the set of Walrasian price vectors is the intersection of the set of effective C-Lindahl price vectors and the 45-degree line, which is the solid blue line in Figure 2.

but not in the other.

7 Conclusions

Studying a general collective choice problem with quasilinear utility, this paper bridges the gap between the classic pricing concept for public goods, Lindahl prices, with the transfers of the VCG mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) that resolve the incentive problem arising from private information. It shows that any agent's preferred, that is, smallest, Lindahl price is equal to his VCG transfer, while the firm's VCG transfer consists of the largest sum of Lindahl prices.

With private goods, the agents' preferred Lindahl prices, and thus their VCG transfers, differ in general from the payments associated with the smallest Walrasian prices because the latter impose the additional restrictions of linearity and anonymity. For the same reason, Walrasian prices may fail to exist whereas Lindahl prices always exist and always support all efficient allocations.

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Appendix A: Proofs

Proof of Proposition 1

1. Toward a contradiction, suppose that there are a Lindahl price vector $\boldsymbol{\lambda} \in \Lambda$ and an efficient allocation $a^\sharp \in \mathcal{A}^*$ such that $\boldsymbol{\lambda}$ does not support $a^\sharp$; that is, either

$$a^\sharp \notin \arg\max_{a \in \mathcal{A}} v_i^a - \lambda_i^a \text{ for some } i \in \mathcal{N} \quad \text{or} \quad a^\sharp \notin \arg\max_{a \in \mathcal{A}} \sum_{i \in \mathcal{N}} \lambda_i^a - c^a.$$

   Then, there exists an allocation $\tilde{a} \in \mathcal{A}$ such that either

$$v_i^{\tilde{a}} - \lambda_i^{\tilde{a}} > v_i^{a^\sharp} - \lambda_i^{a^\sharp} \text{ for some } i \in \mathcal{N} \quad \text{or} \quad \sum_{i \in \mathcal{N}} \lambda_i^{\tilde{a}} - c^{\tilde{a}} > \sum_{i \in \mathcal{N}} \lambda_i^{a^\sharp} - c^{a^\sharp}.$$

   As $\boldsymbol{\lambda}$ is a Lindahl price vector, it supports an efficient allocation $a^* \in \mathcal{A}^*$; which implies that $v_i^{a^*} - \lambda_i^{a^*} \geq v_i^{\tilde{a}} - \lambda_i^{\tilde{a}}$ for every $i \in \mathcal{N}$ and $\sum_{i \in \mathcal{N}} \lambda_i^{a^*} - c^{a^*} \geq \sum_{i \in \mathcal{N}} \lambda_i^{\tilde{a}} - c^{\tilde{a}}$. We therefore have that either

$$v_i^{a^*} - \lambda_i^{a^*} > v_i^{a^\sharp} - \lambda_i^{a^\sharp} \text{ for some } i \in \mathcal{N} \quad \text{or} \quad \sum_{i \in \mathcal{N}} \lambda_i^{a^*} - c^{a^*} > \sum_{i \in \mathcal{N}} \lambda_i^{a^\sharp} - c^{a^\sharp}. \tag{8}$$

   Moreover, as $\boldsymbol{\lambda}$ supports a^*, we have that $v_i^{a^*} - \lambda_i^{a^*} \geq v_i^{a\sharp} - \lambda_i^{a\sharp}$ for every $i \in \mathcal{N}$ and $\sum_{i \in \mathcal{N}} \lambda_i^{a^*} - c^{a^*} \geq \sum_{i \in \mathcal{N}} \lambda_i^{a\sharp} - c^{a\sharp}$. Summing over all agents and using the strict inequality from (8) yields

$$\begin{aligned} \sum_{i \in \mathcal{N}} v_i^{a^*} - \sum_{i \in \mathcal{N}} \lambda_i^{a^*} + \sum_{i \in \mathcal{N}} \lambda_i^{a^*} - c^{a^*} &> \sum_{i \in \mathcal{N}} v_i^{a^\sharp} - \sum_{i \in \mathcal{N}} \lambda_i^{a^\sharp} + \sum_{i \in \mathcal{N}} \lambda_i^{a^\sharp} - c^{a^\sharp} \\ \Leftrightarrow \quad \sum_{i \in \mathcal{N}} v_i^{a^*} - c^{a^*} &> \sum_{i \in \mathcal{N}} v_i^{a^\sharp} - c^{a^\sharp}, \end{aligned}$$

   which contradicts the assumption that $a^\sharp$ is an efficient allocation.

2. We show in turn that Λ is nonempty, closed and bounded.

   Nonempty. We construct a price vector $\boldsymbol{\lambda}$ such that $\lambda_i^a = v_i^a$ for each $i \in \mathcal{N}$ and each $a \in \mathcal{A}$. Note that, by assumption, this vector satisfies the requirement that $\lambda_i^{a_0} = 0$ for every $i \in \mathcal{N}$. We show that $\boldsymbol{\lambda}$ is a Lindahl price vector by showing that it satisfies (1) and (2). The price vector $\boldsymbol{\lambda}$ satisfies (1) since, for any agent $i \in \mathcal{N}$, any efficient allocation $a^* \in \mathcal{A}$, and any allocation $a \in \mathcal{A}$, we have that

$$v_i^{a^*} - \lambda_i^{a^*} = 0 = v_i^a - \lambda_i^a.$$

The price $\boldsymbol{\lambda}$ satisfies (2) since, for any efficient allocation $a^* \in \mathcal{A}$ and any allocation $a \in \mathcal{A}$, we have that

$$\sum_{i \in \mathcal{N}} \lambda_i^{a^*} - c^{a^*} = \sum_{i \in \mathcal{N}} v_i^{a^*} - c^{a^*} = W \geq \sum_{i \in \mathcal{N}} v_i^a - c^a = \sum_{i \in \mathcal{N}} \lambda_i^a - c^a,$$

where the inequality stems from the fact that no allocation can create a larger level of social welfare than the efficient level. We conclude that $\boldsymbol{\lambda}$ is a Lindahl price vector, hence such a vector exists and $\Lambda \neq \varnothing$.

Closed. Consider any convergent sequence of Lindahl price vectors $(\boldsymbol{\lambda}(n))_{n \in \mathbb{N}}$ and let $\boldsymbol{\lambda} = \lim_{n \to \infty} \boldsymbol{\lambda}(n)$. Towards a contradiction, suppose that $\boldsymbol{\lambda} \notin \Lambda$. Then, for any efficient alternative $a^* \in \mathcal{A}^*$, there exists an alternative $a \in \mathcal{A}$ such that

$$\text{either } v_i^{a^*} - \lambda_i^{a^*} < v_i^a - \lambda_i^a \text{ for some } i \in \mathcal{N} \text{ or } \sum_{i \in \mathcal{N}} \lambda_i^{a^*} - c^{a^*} < \sum_{i \in \mathcal{N}} \lambda_i^a - c^a.$$

As the sequence converges to $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}_n$ is arbitrarily close to $\boldsymbol{\lambda}$ for a large enough n. Therefore, there exists $n \in \mathbb{N}$ such that

$$\text{either } v_i^{a^*} - \lambda_i^{a^*}(n) < v_i^a - \lambda_i^a(n) \text{ for some } i \in \mathcal{N} \text{ or } \sum_{i \in \mathcal{N}} \lambda_i^{a^*}(n) - c^{a^*} < \sum_{i \in \mathcal{N}} \lambda_i^a(n) - c^a.$$

It follows that $\boldsymbol{\lambda}(n) \notin \Lambda$, a contradiction. We conclude that every convergent sequence of Lindahl price vector converges to a Lindahl price vector, hence Λ is closed.

Bounded. For any Lindahl price vector $\boldsymbol{\lambda}$, any alternative $a \in \mathcal{A}^*$ and any efficient alternative $a^* \in \mathcal{A}^*$, we have that

$$\sum_{i \in \mathcal{N}} \lambda_i^{a^*} - c^{a^*} \geq \sum_{i \in \mathcal{N}} \lambda_i^a - c^a \quad \Leftrightarrow \quad \sum_{i \in \mathcal{N}} \lambda_i^a \leq \sum_{i \in \mathcal{N}} \lambda_i^{a^*} - c^{a^*} + c^a$$

since the firm must pick a^* over a. Moreover, $\lambda_i^{a^*} \leq v_i^{a^*}$ for every $i \in \mathcal{N}$ since each agent must pick a^* over the null. Combining our two inequalities and defining $c^{\max} = \max_{a \in \mathcal{A}} c^a$ yields

$$\sum_{i \in \mathcal{N}} \lambda_i^a \leq \sum_{i \in \mathcal{N}} v_i^{a^*} - c^{a^*} + c^a = W + c^a \leq W + c^{\max}.$$

As Lindahl prices are nonnegative, no term can exceed the sum so we have that $\lambda_i^a \leq W + c^{\max}$ for every agent $i \in \mathcal{N}$ and every alternative $a \in \mathcal{A}$. It follows that the length of every Lindahl price vector $\boldsymbol{\lambda}$ is bounded by

$$\sqrt{\sum_{i\in\mathcal{N}}\sum_{a\in\mathcal{A}}(W + c^{\max})^2} = \sqrt{NA(W + c^{\max})^2} = \sqrt{NA}(W + c^{\max}).$$

By definition, Λ is bounded since there is an upper bound on the length of all of its elements. $\square$

Proof of Theorem 1

Agents. ($\underline{\lambda}_i^{a^*} \geq \underline{\tau}_i^{a^*}$) Consider any agent $i \in \mathcal{N}$ and any Lindahl price vector $\boldsymbol{\lambda} \in \Lambda$. We need to show that $\lambda_i^{a^*} \geq \underline{\tau}_i^{a^*}$. By (1), for every $j \in \mathcal{N} \setminus \{i\}$, we have that

$$\begin{aligned} & v_j^{a^*} - \lambda_j^{a^*} \geq v_j^{a^*_{-i}} - \lambda_j^{a^*_{-i}} \\ \Leftrightarrow \quad & \lambda_j^{a^*} \leq v_j^{a^*} - v_j^{a^*_{-i}} + \lambda_j^{a^*_{-i}}. \end{aligned}$$

Summing over all agents other than i yields:

$$\sum_{j\in\mathcal{N}\setminus\{i\}} \lambda_j^{a^*} \leq \sum_{j\in\mathcal{N}\setminus\{i\}} v_j^{a^*} - \sum_{j\in\mathcal{N}\setminus\{i\}} v_j^{a^*_{-i}} + \sum_{j\in\mathcal{N}\setminus\{i\}} \lambda_j^{a^*_{-i}}. \tag{9}$$

By (2), we have that

$$\begin{aligned} & \sum_{j\in\mathcal{N}} \lambda_j^{a^*} - c^{a^*} \geq \sum_{j\in\mathcal{N}} \lambda_j^{a^*_{-i}} - c^{a^*_{-i}} \\ \Leftrightarrow \quad & \lambda_i^{a^*} \geq \sum_{j\in\mathcal{N}} \lambda_j^{a^*_{-i}} - c^{a^*_{-i}} + c^{a^*} - \sum_{j\in\mathcal{N}\setminus\{i\}} \lambda_j^{a^*}. \end{aligned} \tag{10}$$

Combining (9) with (10) yields

$$\begin{aligned} \lambda_i^{a^*} &\geq \sum_{j\in\mathcal{N}} \lambda_i^{a^*_{-i}} - c^{a^*_{-i}} + c^{a^*} - \sum_{j\in\mathcal{N}\setminus\{i\}} v_j^{a^*} + \sum_{j\in\mathcal{N}\setminus\{i\}} v_j^{a^*_{-i}} - \sum_{j\in\mathcal{N}\setminus\{i\}} \lambda_j^{a^*_{-i}} \\ &= \underbrace{\sum_{j\in\mathcal{N}\setminus\{i\}} v_j^{a^*_{-i}} - c^{a^*_{-i}} - \left(\sum_{j\in\mathcal{N}\setminus\{i\}} v_j^{a^*} - c^{a^*}\right)}_{=\underline{\tau}_i^{a^*}} + \underbrace{\lambda_i^{a^*_{-i}}}_{\geq 0} \geq \underline{\tau}_i^{a^*}. \end{aligned}$$

($\underline{\lambda}_i^{a^*} \leq \underline{\tau}_i^{a^*}$) Consider any agent $i \in \mathcal{N}$. We need to show that there exists a Lindahl price vector $\boldsymbol{\lambda} \in \Lambda$ such that $\lambda_i^{a^*} = \underline{\tau}_i^{a^*}$. We construct $\boldsymbol{\lambda}$ as follows:

- Set $\lambda_i^{a^*} = \underline{\tau}_i^{a^*}$

- For every $a \in \mathcal{A} \setminus \{a^*\}$, set $\lambda_i^a = \max\{v_i^a - (W - W_{-i}), 0\}$.

- For every $j \in \mathcal{N} \setminus \{i\}$ and every $a \in \mathcal{A}$, set $\lambda_j^a = v_j^a$.

We need to show that $\boldsymbol{\lambda}$ satisfies (1) and (2). For any agent $j \in \mathcal{N} \setminus \{i\}$ and any alternative $a \in \mathcal{A} \setminus \{a^*\}$, we have that

$$v_j^{a^*} - \lambda_j^{a^*} = 0 = v_j^a - \lambda_j^a,$$

hence $\boldsymbol{\lambda}$ satisfies (1) for all agents other than i. For agent i, we have that

$$v_i^{a^*} - \lambda_i^{a^*} = v_i^{a^*} - \underline{\tau}_i^{a^*} = v_i^{a^*} - W_{-i} + W - v_i^{a^*} = W - W_{-i}.$$

Moreover, for any alternative $a \in \mathcal{A} \setminus \{a^*\}$, we have that

$$v_i^a - \lambda_i^a = v_i^a - \max\{v_i^a - (W - W_{-i}), 0\} \leq v_i^a - v_i^a + (W - W_{-i}) = W - W_{-i}.$$

We conclude that $v_i^{a^*} - \lambda_i^{a^*} \geq v_i^a - \lambda_i^a$ for every $a \in \mathcal{A} \setminus \{a^*\}$, hence $\boldsymbol{\lambda}$ satisfies (1) for i as well.

Turning to the firm and to condition (2), we need to show that, for every allocation $a \in \mathcal{A} \setminus \{a^*\}$,

$$\begin{aligned}
& \sum_{j \in \mathcal{N}} \lambda_j^{a^*} - c^{a^*} \geq \sum_{j \in \mathcal{N}} \lambda_j^a - c^a \\
\Leftrightarrow \quad & \underline{\tau}_i^{a^*} + \sum_{j \in \mathcal{N} \setminus \{i\}} \lambda_j^{a^*} - c^{a^*} \geq \lambda_i^a + \sum_{j \in \mathcal{N} \setminus \{i\}} \lambda_j^a - c^a \\
\Leftrightarrow \quad & W_{-i} - W + \underbrace{v_i^{a^*} + \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{a^*} - c^{a^*}}_{=W} \geq \max\{v_i^a - (W - W_{-i}), 0\} + \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^a - c^a \\
\Leftrightarrow \quad & W_{-i} \geq \max\{v_i^a - (W - W_{-i}), 0\} + \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^a - c^a.
\end{aligned}$$

We separately consider two cases depending on whether $v_i^a \geq W - W_{-i}$ or $v_i^a < W - W_{-i}$.

Case 1: $v_i^a \geq W - W_{-i}$. In that case, $\lambda_i^a = v_i^a - (W - W_{-i})$ so we need to show that

$$\begin{aligned}
& W_{-i} \geq v_i^a - (W - W_{-i}) + \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^a - c^a \\
\Leftrightarrow \quad & W \geq \sum_{j \in \mathcal{N}} v_j^a - c^a,
\end{aligned}$$

which is satisfied since, in the economy where i is present, no alternative can provide a larger social welfare than the efficient level.

Case 2: $v_i^a < W - W_{-i}$. In that case, $\lambda_i^a = 0$ so we need to show that

$$W_{-i} \geq \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^a - c^a,$$

which is satisfied since, in the economy where i is absent, no alternative can provide a larger social welfare than the efficient level.

Firm. ($\overline{\lambda}_f^{a^*} \leq \overline{\tau}_f^{a^*}$) Consider any Lindahl price vector $\boldsymbol{\lambda} \in \Lambda$. We need to show that $\sum_{i \in \mathcal{N}} \lambda_i^{a^*} \leq \overline{\tau}_f^{a^*}$. By (1), $\boldsymbol{\lambda}$ incentivizes each agent to pick a^* over the null; hence, $v_i^{a^*} - \lambda_i^{a^*} \geq 0$, which is equivalent to $\lambda_i^{a^*} \leq v_i^{a^*}$ and implies that

$$\sum_{i \in \mathcal{N}} \lambda_i^{a^*} \leq \sum_{i \in \mathcal{N}} v_i^{a^*} = \overline{\tau}_f^{a^*}.$$

($\overline{\lambda}_f^{a^*} \geq \overline{\tau}_f^{a^*}$) We need to show that there exists a Lindahl price vector $\boldsymbol{\lambda} \in \Lambda$ such that $\sum_{i \in \mathcal{N}} \lambda_i^{a^*} = \overline{\tau}_f^{a^*}$. We construct $\boldsymbol{\lambda}$ as follows: For every $i \in \mathcal{N}$ and every $a \in \mathcal{A}$, set $\lambda_i^a = v_i^a$. It is immediate that $\sum_{i \in \mathcal{N}} \lambda_i^{a^*} = \sum_{i \in \mathcal{N}} v_i^{a^*} = \overline{\tau}_f^{a^*}$; therefore, it remains to show that $\boldsymbol{\lambda}$ satisfies (1) and (2). For every $i \in \mathcal{N}$ and every $a \in \mathcal{A} \setminus \{a^*\}$, we have that $v_i^{a^*} - \lambda_i^{a^*} = 0 = v_i^a - \lambda_i^a$ so $\boldsymbol{\lambda}$ satisfies (1). Turning to the firm and condition (2), for every $a \in \mathcal{A} \setminus \{a^*\}$, we have that

$$\sum_{i \in \mathcal{N}} \lambda_i^{a^*} - c^{a^*} = \sum_{i \in \mathcal{N}} v_i^{a^*} - c^{a^*} \geq \sum_{i \in \mathcal{N}} v_i^a - c^a = \sum_{i \in \mathcal{N}} \lambda_i^a - c^a,$$

where each equality follows from the construction of $\boldsymbol{\lambda}$ and the inequality follows from the efficiency of a^*. We conclude that $\boldsymbol{\lambda}$ satisfies (2). $\square$

Proof of Theorem 2

Agents. Consider any agent $i \in \mathcal{N}$. Since $\Lambda(\mathcal{P}) = \Lambda \cap \mathcal{P}$, it is $\Lambda(\mathcal{P}) \subseteq \Lambda$ so $\underline{\lambda}_i^{a^*}(\mathcal{P}) = \min_{\boldsymbol{\lambda} \in \Lambda(\mathcal{P})} \lambda_i^{a^*} \geq \min_{\boldsymbol{\lambda} \in \Lambda} \lambda_i^{a^*} = \underline{\lambda}_i^{a^*}$. Combining this result with Theorem 1 yields $\underline{\lambda}_i^{a^*}(\mathcal{P}) \geq \underline{\lambda}_i^{a^*} = \underline{\tau}_i^{a^*}$. If $(\arg\min_{\boldsymbol{\lambda} \in \Lambda} \lambda_i^{a^*}) \cap \mathcal{P} \neq \varnothing$, then $\underline{\lambda}_i^{a^*}(\mathcal{P}) = \min_{\boldsymbol{\lambda} \in \Lambda(\mathcal{P})} \lambda_i^{a^*} = \min_{\boldsymbol{\lambda} \in \Lambda} \lambda_i^{a^*} = \underline{\lambda}_i^{a^*}$ and Theorem 1 yields $\underline{\lambda}_i^{a^*}(\mathcal{P}) = \underline{\lambda}_i^{a^*} = \underline{\tau}_i^{a^*}$. If $(\arg\min_{\boldsymbol{\lambda} \in \Lambda} \lambda_i^{a^*}) \cap \mathcal{P} = \varnothing$, then either $\Lambda(\mathcal{P}) = \varnothing$ and $\underline{\lambda}_i^{a^*}$ is not defined or $(\arg\min_{\boldsymbol{\lambda} \in \Lambda} \lambda_i^{a^*}) \cap \mathcal{P} \neq \varnothing$ and $\underline{\lambda}_i^{a^*}(\mathcal{P}) = \min_{\boldsymbol{\lambda} \in \Lambda(\mathcal{P})} \lambda_i^{a^*} > \min_{\boldsymbol{\lambda} \in \Lambda} \lambda_i^{a^*} = \underline{\lambda}_i^{a^*}$ and Theorem 1 yields $\underline{\lambda}_i^{a^*}(\mathcal{P}) > \underline{\lambda}_i^{a^*} = \underline{\tau}_i^{a^*}$.

Firm. $\Lambda(\mathcal{P}) = \Lambda \cap \mathcal{P}$ implies that $\overline{\lambda}_f^{a^*}(\mathcal{P}) = \max_{\boldsymbol{\lambda}\in\Lambda(\mathcal{P})} \sum_{i\in\mathcal{N}} \lambda_i^{a^*} \leq \max_{\boldsymbol{\lambda}\in\Lambda} \sum_{i\in\mathcal{N}} \lambda_i^{a^*} = \overline{\lambda}_f^{a^*}$, which combined with Theorem 1 yields $\overline{\lambda}_f^{a^*}(\mathcal{P}) \leq \overline{\lambda}_f^{a^*} = \overline{\tau}_f^{a^*}$. If $(\arg\max_{\boldsymbol{\lambda}\in\Lambda} \sum_{i\in\mathcal{N}} \lambda_i^{a^*}) \cap \mathcal{P} \neq \varnothing$, then $\overline{\lambda}_f^{a^*}(\mathcal{P}) = \max_{\boldsymbol{\lambda}\in\Lambda(\mathcal{P})} \sum_{i\in\mathcal{N}} \lambda_i^{a^*} = \max_{\boldsymbol{\lambda}\in\Lambda} \sum_{i\in\mathcal{N}} \lambda_i^{a^*} = \overline{\lambda}_f^{a^*}$ and Theorem 1 yields $\overline{\lambda}_f^{a^*}(\mathcal{P}) = \overline{\lambda}_f^{a^*} = \overline{\tau}_f^{a^*}$. If $(\arg\max_{\boldsymbol{\lambda}\in\Lambda} \sum_{i\in\mathcal{N}} \lambda_i^{a^*}) \cap \mathcal{P} = \varnothing$, then either $\Lambda(\mathcal{P}) = \varnothing$ and $\overline{\lambda}_f^{a^*}$ is not defined or $(\arg\max_{\boldsymbol{\lambda}\in\Lambda} \sum_{i\in\mathcal{N}} \lambda_i^{a^*}) \cap \mathcal{P} \neq \varnothing$ and $\overline{\lambda}_f^{a^*}(\mathcal{P}) = \max_{\boldsymbol{\lambda}\in\Lambda(\mathcal{P})} \sum_{i\in\mathcal{N}} \lambda_i^{a^*} < \max_{\boldsymbol{\lambda}\in\Lambda} \sum_{i\in\mathcal{N}} \lambda_i^{a^*} = \overline{\lambda}_f^{a^*}$ and Theorem 1 yields $\overline{\lambda}_f^{a^*}(\mathcal{P}) < \overline{\lambda}_f^{a^*} = \overline{\tau}_f^{a^*}$. □

Proof of Proposition 2

If. Suppose that $\boldsymbol{\mu}$ is anonymous and linear. Pick any agent $j \in \mathcal{N}$ and construct the object price vector $\boldsymbol{p} \in \mathbb{R}^O_{\geq 0}$ by setting $p_o = \mu_j^{a_j^o}$ for each $o \in \mathcal{O}$. Consider next any agent $i \in \mathcal{N}$ and any alternative $a \in \mathcal{A}$. As $\boldsymbol{\mu}$ is anonymous, we have that $\mu_i^{a_i^o} = \mu_j^{a_j^o} = p_o$ (since $B_i^{a_i^o} = B_j^{a_j^o} = \{o\}$). As $\boldsymbol{\mu}$ is linear, we have that $\mu_i^a = \sum_{o\in B_i^a} \mu_i^{a_i^o} = \sum_{o\in B_i^a} p_o$. As this holds for all agents and all alternatives, we conclude that $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{p})$, and therefore $\boldsymbol{\mu} \in \mathcal{W}$.

Only if. Suppose that $\boldsymbol{\mu} \in \mathcal{W}$. Then, there exists an object price vector $\boldsymbol{p} \in \mathbb{R}^O$ such that $\mu_i^a = \sum_{o\in B_i^o} p_o$ for every $i \in \mathcal{N}$ and every $a \in \mathcal{A}$. For any $i, j \in \mathcal{N}$ and any $a, a' \in \mathcal{A}$ such that $B_i^a = B_j^{a'}$, we have that $\mu_i^a = \sum_{o\in B_i^a} p_o = \sum_{o\in B_j^{a'}} p_o = \mu_j^{a'}$; therefore, $\boldsymbol{\mu}$ is anonymous. For any $i \in \mathcal{N}$ and any $o \in O$, $\mu_i^{a_i^o} = \sum_{o'\in B_i^{a_i^o}} p'_o = p_o$; therefore, for any $a \in \mathcal{A}$, $\mu_i^a = \sum_{o\in B_i^a} p_o = \sum_{o\in B_i^a} \mu^{a_i^o}$ and $\boldsymbol{\mu}$ is linear. □

Proof of Proposition 3

If. Suppose that $|L^{a^*}| = 1$ and denote by $\boldsymbol{\ell}^{a^*}$ the unique effective Lindahl price vector, i.e., $L^{a^*} = \{\boldsymbol{\ell}^{a^*}\}$. For each agent $i \in \mathcal{N}$, we have that

$$\underbrace{\underline{\tau}_i^{a^*} = \underline{\lambda}_i^{a^*}}_{\text{By Theorem 1}} \underbrace{= \min_{\boldsymbol{\lambda}\in\Lambda} \lambda_i^{a^*}}_{\text{By definition}} \underbrace{= \min_{\boldsymbol{\ell}\in L^{a^*}} \ell_i}_{\text{By definition}} \underbrace{= \ell_i^{a^*}}_{\text{As } L^{a^*}=\{\boldsymbol{\ell}^{a^*}\}}.$$

For the firm, we have that

$$\underbrace{\overline{\tau}_f^{a^*} = \overline{\lambda}_f^{a^*}}_{\text{By Theorem 1}} \underbrace{= \max_{\boldsymbol{\lambda}\in\Lambda} \sum_{i\in\mathcal{N}} \lambda_i^{a^*}}_{\text{By definition}} \underbrace{= \max_{\boldsymbol{\ell}\in L^{a^*}} \sum_{i\in\mathcal{N}} \ell_i}_{\text{By definition}} \underbrace{= \sum_{i\in\mathcal{N}} \ell_i^{a^*}}_{\text{As } L^{a^*}=\{\boldsymbol{\ell}^{a^*}\}}.$$

We conclude that the VCG deficit is

$$D^{VCG} = \overline{\tau}_f^{a^*} - \sum_{i\in\mathcal{N}} \underline{\tau}_i^{a^*} = \sum_{i\in\mathcal{N}} \ell_i^{a^*} - \sum_{i\in\mathcal{N}} \ell_i^{a^*} = 0.$$

Only If. We need to show that $D^{VCG} = 0$ implies $|L^{a^*}| = 1$. We prove the contrapositive: $|L^{a^*}| \neq 1$ implies $D^{VCG} \neq 0$. As the set of Lindahl price vectors is nonempty (Proposition 1), so is the set of effective Lindahl price vectors, hence $|L^{a^*}| \neq 1$ is equivalent to $|L^{a^*}| > 1$. Moreover, as by (5) the VCG deficit is nonnegative, $D^{VCG} \neq 0$ is equivalent to $D^{VCG} > 0$. We therefore assume that $|L^{a^*}| > 1$ and show that $D^{VCG} > 0$. We consider two cases separately depending on whether or not there exists a vector that is most favorable to all agents.

Case 1: There exists $\tilde{\boldsymbol{\ell}} \in L^{a^*}$ such that, for every $i \in \mathcal{N}$ and every $\boldsymbol{\ell} \in L^{a^*}$, $\tilde{\ell}_i \leq \ell_i$. Each agent i's VCG transfer is

$$\underbrace{\underline{\tau}_i^{a^*} =}_{\text{By Theorem 1}} \underbrace{\underline{\lambda}_i^{a^*} =}_{\text{By definition}} \underbrace{\min_{\boldsymbol{\lambda} \in \Lambda} \lambda_i^{a^*} =}_{\text{By definition}} \underbrace{\min_{\boldsymbol{\ell} \in L^{a^*}} \ell_i \quad =}_{\text{By case assumption}} \tilde{\ell}_i \quad .$$

As $|L^{a^*}| > 1$, there exists an effective Lindahl price vector $\boldsymbol{\ell}' \neq \tilde{\boldsymbol{\ell}}$. Then, the case assumption implies that $\ell_i' \geq \tilde{\ell}_i$ for all $i \in \mathcal{N}$ and $\ell_j' > \tilde{\ell}_j$ for some $j \in \mathcal{N}$; hence,

$$\sum_{i \in \mathcal{N}} \ell_i' > \sum_{i \in \mathcal{N}} \tilde{\ell}_i. \tag{11}$$

It follows that

$$\underbrace{\overline{\tau}_f^{a^*} =}_{\text{By Theorem 1}} \underbrace{\overline{\lambda}_f^{a^*} =}_{\text{By definition}} \underbrace{\max_{\boldsymbol{\lambda} \in \Lambda} \sum_{i \in \mathcal{N}} \lambda_i^{a^*} =}_{\text{By definition}} \underbrace{\max_{\boldsymbol{\ell} \in L^{a^*}} \sum_{i \in \mathcal{N}} \ell_i \geq}_{\text{By definition}} \underbrace{\sum_{i \in \mathcal{N}} \ell_i' >}_{\text{By (11)}} \sum_{i \in \mathcal{N}} \tilde{\ell}_i \, .$$

We conclude that the VCG deficit is

$$D^{VCG} = \overline{\tau}_f^{a^*} - \sum_{i \in \mathcal{N}} \underline{\tau}_i^{a^*} > \sum_{i \in \mathcal{N}} \tilde{\ell}_i - \sum_{i \in \mathcal{N}} \tilde{\ell}_i = 0.$$

Case 2: For every $\boldsymbol{\ell} \in L^{a^*}$, there exists $j \in \mathcal{N}$ and $\boldsymbol{\ell}' \in L^{a^*}$ such that $\ell_j' < \ell_j$. Denote by $\hat{\boldsymbol{\ell}}$ the effective Lindahl price vector that maximizes the firm's revenue. We have that

$$\underbrace{\overline{\tau}_f^{a^*} =}_{\text{By Theorem 1}} \underbrace{\overline{\lambda}_f^{a^*} =}_{\text{By definition}} \underbrace{\max_{\boldsymbol{\lambda} \in \Lambda} \sum_{i \in \mathcal{N}} \lambda_i^{a^*} =}_{\text{By definition}} \underbrace{\max_{\boldsymbol{\ell} \in L^{a^*}} \sum_{i \in \mathcal{N}} \ell_i =}_{\text{By definition}} \sum_{i \in \mathcal{N}} \hat{\ell}_i \, .$$

By definition, for every $i \in \mathcal{N}$, $\min_{\boldsymbol{\ell} \in L^{a^*}} \ell_i \leq \hat{\ell}_i$. Moreover, by the case assumption, there exists an agent $j \in \mathcal{N}$ such that $\min_{\boldsymbol{\ell} \in L^{a^*}} \ell_j < \hat{\ell}_j$. It follows that

$$\sum_{i \in \mathcal{N}} \min_{\boldsymbol{\ell} \in L^{a^*}} \ell_i < \sum_{i \in \mathcal{N}} \hat{\ell}_i. \tag{12}$$

It follows that

$$\sum_{i\in\mathcal{N}} \underline{\tau}_i^{a^*} \underbrace{=\sum_{i\in\mathcal{N}} \underline{\lambda}_i^{a^*}}_{\text{By Theorem 1}} \underbrace{=\sum_{i\in\mathcal{N}} \min_{\boldsymbol{\lambda}\in\Lambda} \lambda_i^{a^*}}_{\text{By definition}} \underbrace{=\sum_{i\in\mathcal{N}} \min_{\boldsymbol{\ell}\in L^{a^*}} \ell_i}_{\text{By definition}} \underbrace{<\sum_{i\in\mathcal{N}} \hat{\ell}_i}_{\text{By (12)}}.$$

We conclude that the VCG deficit is

$$D^{VCG} = \overline{\tau}_f^{a^*} - \sum_{i\in\mathcal{N}} \underline{\tau}_i^{a^*} > \sum_{i\in\mathcal{N}} \hat{\ell}_i - \sum_{i\in\mathcal{N}} \hat{\ell}_i = 0.$$

□

Proof of Proposition 4

Only if. Suppose $\underline{\boldsymbol{\lambda}}^{a^*} \in L_C^{a^*}$, then we have that:

$$\sum_{i\in\mathcal{N}} \underline{\tau}_i^{a^*} \underbrace{=\sum_{i\in\mathcal{N}} \underline{\lambda}_i^{a^*}}_{\text{By Theorem 1}} \underbrace{\geq \quad c^{a^*}}_{\text{As } \underline{\boldsymbol{\lambda}}^{a^*}\in L_C^{a^*}}.$$

If. Suppose $\underline{\boldsymbol{\lambda}}^{a^*} \notin L_C^{a^*}$. We consider two cases.

Case 1: $a^* = a_0$. Then, $L_C^{a^*} = L_C^{a_0} = \{(0)_{i\in\mathcal{N}}\}$ since the price of the null is zero for all agents. It follows that $\underline{\boldsymbol{\lambda}}^{a^*} = (\underline{\lambda}_i^{a^*})_{i\in\mathcal{N}} = (0)_{i\in\mathcal{N}} \in L_C^{a^*}$, which contradicts our assumption.

Case 2: $a^* \neq a_0$. Construct a price vector $\boldsymbol{\lambda}$ such that:

- $\lambda_i^{a^*} = \underline{\lambda}_i^{a^*}$ for every $i \in \mathcal{N}$; and
- $\lambda_i^a = v_i^a$ for every $i \in \mathcal{N}$ and every $a \in \mathcal{A} \setminus \{a^*\}$.

Consider any agent i. By definition, there exists a Lindahl price vector in which i's price for a^* is $\underline{\lambda}_i^{a^*}$; therefore, $v_i^{a^*} - \underline{\lambda}_i^{a^*} \geq 0$ (as a Lindahl price vector incentivizes i to pick a^* over the null). By construction, for every $a \in \mathcal{A} \setminus \{a^*\}$, $v_i^a - \lambda_i^a = 0$ so $v_i^{a^*} - \underline{\lambda}_i^{a^*} \geq v_i^a - \lambda_i^a$. Therefore, $\boldsymbol{\lambda}$ satisfies condition (1) for every agent.

Note that if the constructed price vector $\boldsymbol{\lambda}$ satisfied $\boldsymbol{\lambda} \in \Lambda_C^{a^*}$, then it would be $\underline{\boldsymbol{\lambda}}^{a^*} \in L_C^{a^*}$. Since, by assumption, $\underline{\boldsymbol{\lambda}}^{a^*} \notin L_C^{a^*}$, it must be $\boldsymbol{\lambda} \notin \Lambda_C^{a^*}$. Then, by definition, $\boldsymbol{\lambda}$ must violate (7); that is, we have that $\sum_{i\in\mathcal{N}} \underline{\lambda}_i^{a^*} = \sum_{i\in\mathcal{N}} \lambda_i^{a^*} < c^{a^*}$, which by Theorem 1 implies that $\sum_{i\in\mathcal{N}} \underline{\tau}_i^{a^*} < c^{a^*}$. □

Appendix B: Effective Lindahl Prices and Marginal Core

In this appendix, we connect effective Lindahl price vectors with the marginal core defined by Segal and Whinston (2016). Translated to our setting, Segal and Whinston's (2016) Proposition 1 states that *if* the set of effective marginal core price vectors is nonempty for all type profiles and multi-valued for at least one type profile that is drawn with positive probability, then the expected VCG deficit is positive. In our setting, the set of effective marginal core price vectors is nonempty for all types. We show that the VCG deficit is positive for a type profile *if and only if* that type profile yields a multi-valued set of effective marginal core price vectors.

Define an *effective price vector* to be a nonnegative N-dimensional vector that specifies a price for each agent (i.e., the set of all effective price vectors is $\mathbb{R}^N_{\geq 0}$). Recall from Section 6 that, for each efficient allocation $a^* \in \mathcal{A}^*$, the set of effective Lindahl price vectors L^{a^*} contains all effective Lindahl price vectors whose prices for a^* are the same as those of a Lindahl price vector: $L^{a^*} = \{\boldsymbol{\ell} \in \mathbb{R}^N_{\geq 0} : \boldsymbol{\ell} = (\lambda_i^{a^*})_{i \in \mathcal{N}} \text{ for some } \boldsymbol{\lambda} \in \Lambda\}$.

Expressing Segal and Whinston's (2016) definition in our terminology and notation, an effective price vector $\mathbf{m} \in \mathbb{R}^N_{\geq 0}$ is an *effective marginal core price vector* for efficient allocation $a^* \in \mathcal{A}^*$ if the payoffs generated by $\mathbf{m}$ at a^* are such that no coalition of all but one participants can gain by deviating. Formally, denoting the set of effective marginal core price vector for efficient allocation $a^* \in \mathcal{A}^*$ by M^{a^*}, we have that $\mathbf{m} \in M^{a^*}$ if

$$\sum_{j \in \mathcal{N} \setminus \{i\}} \left(v_j^{a^*} - m_j\right) + \sum_{j \in \mathcal{N}} m_j - c^{a^*} \geq W_{-i} \quad \text{for every } i \in \mathcal{N} \quad \text{and} \tag{13}$$

$$\sum_{j \in \mathcal{N}} \left(v_j^{a^*} - m_j\right) \geq W_{-f} = 0 \tag{14}$$

For each $i \in \mathcal{N}$, condition (13) specifies that the coalition including the firm and all agents except i cannot gain by deviating. Condition (14) specifies the same condition for the coalition formed of all agents (without the firm). We can rearrange (13) as follows:

$$
\begin{aligned}
& \sum_{j\in\mathcal{N}\setminus\{i\}} \left(v_j^{a^*} - m_j\right) + \sum_{j\in\mathcal{N}} m_j - c^{a^*} \geq W_{-i} \\
\Leftrightarrow \quad & \underbrace{\sum_{j\in\mathcal{N}\setminus\{i\}} v_j^{a^*} - c^{a^*}}_{=W-v_i^{a^*}} + \underbrace{\sum_{j\in\mathcal{N}} m_j - \sum_{j\in\mathcal{N}\setminus\{i\}} m_j}_{=m_i} \geq W_{-i} \\
\Leftrightarrow \quad & m_i \geq W_{-i} - (W - v_i^{a^*}) = \underline{\tau}_i^{a^*}.
\end{aligned}
$$

Therefore, condition (13) requires that each agent's price be greater than or equal to his VCG transfer. Meanwhile, condition (14) requires that the prices be small enough so that the sum of the agents' payoffs under those prices is nonnegative.

An effective Lindahl price vector satisfies (13) since, by Theorem 1, each agent's effective Lindahl price cannot be smaller than his VCG transfer and (14) since each agent's effective Lindahl price cannot exceed his value for the efficient allocation, else the agent's payoff for that allocation would be negative (hence less than the payoff of the null allocation). It follows that every effective Lindahl price vector is an effective marginal core price vector: $L^{a^*} \subseteq M^{a^*}$ for every $a^* \in \mathcal{A}^*$. We next present the main result from this appendix.

Proposition 5. *For every efficient alternative $a^* \in \mathcal{A}^*$, $|L^{a^*}| = 1$ if and only if $|M^{a^*}| = 1$.*

Combining Proposition 5 with Proposition 3, we obtain that $D^{VCG} = 0$ if and only if $|M^{a^*}| = 1$. As L^{a^*} is nonempty, we have that M^{a^*} is nonempty (since $L^{a^*} \subseteq M^{a^*}$); therefore, since the VCG deficit is nonnegative (equation (5)), Propositions 3 and 5 imply that the following three statements are equivalent: (*i*) $D^{VCG} > 0$; (*ii*) $|L^{a^*}| > 1$; (*iii*) $|M^{a^*}| > 1$. Finally, Proposition 5 and the property that $L^{a^*} \subseteq M^{a^*}$ imply the following corollary.

Corollary 1 (to Proposition 5)**.** *If $|M^{a^*}| = 1$ (or, equivalently, $|L^{a^*}| = 1$, or $D^{VCG} = 0$), then $L^{a^*} = M^{a^*}$.*

Proof of Proposition 5

If. Suppose that $|M^{a^*}| = 1$. As $L^{a^*} \neq \varnothing$ (by Proposition 1) and $L^{a^*} \subseteq M^{a^*}$, we have that $|L^{a^*}| = 1$.

Only if. Suppose that $|L^{a^*}| = 1$ and consider any $\mathbf{m} \in M^{a^*}$ (such a vector exists since $\varnothing \neq L^{a^*} \subseteq M^{a^*}$). By (14), $\sum_{i\in\mathcal{N}} \left(v_i^{a^*} - m_i\right) \geq 0$ so $\sum_{i\in\mathcal{N}} m_i \leq \sum_{i\in\mathcal{N}} v_i^{a^*}$. As $|L^{a^*}| = 1$,

$D^{VCG} = 0$ by Proposition 3; hence, $\sum_{i \in \mathcal{N}} (W - W_{-i}) = 0$ by (5). It follows that

$$\sum_{i \in \mathcal{N}} m_i \leq \sum_{i \in \mathcal{N}} v_i^{a^*} - \sum_{i \in \mathcal{N}} (W - W_{-i}) = \sum_{i \in \mathcal{N}} \big(W_{-i} - (W - v_i^{a^*})\big) = \sum_{i \in \mathcal{N}} \underline{\tau}_i^{a^*}. \tag{15}$$

By (13), $m_i \geq \underline{\tau}_i^{a^*}$ for every $i \in \mathcal{N}$. If $m_i > \underline{\tau}_i^{a^*}$ for some $i \in \mathcal{N}$, we therefore have that $\sum_{i \in \mathcal{N}} m_i > \sum_{i \in \mathcal{N}} \underline{\tau}_i^{a^*}$, which contradicts (15). We conclude that $m_i = \underline{\tau}_i^{a^*}$ for every $i \in \mathcal{N}$, hence $(\underline{\tau}_i^{a^*})$ is the unique effective marginal core price vector. $\square$