

Helicopter Drop, Redistribution, and Welfare in Decentralized Monetary Economy

Kazuya Kamiya* and So Kubota[†]

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Abstract

What are the distributional effects of monetary policy in a decentralized economy? This paper develops a tractable model of monetary exchange where agents trade exclusively through bilateral search and Nash bargaining. Despite the money-holdings distribution being non-degenerate and evolving endogenously over time, we derive closed-form solutions for the stationary equilibrium and for an exact on-path transition after a one-time injection. We study helicopter drops in the original sense: a central bank injects money to agents holding heterogeneous balances. Under a long-run constant money growth rule, stationary welfare is strictly increasing throughout the range of positive growth rates for which the stationary equilibrium exists. For the one-time injection, output and welfare rise under an explicit condition and the price level adjusts only gradually. These results are driven by a redistribution channel operating through the endogenous money-holdings distribution and its interaction with the bargaining surplus.

Keywords: Money, Search, Bargaining, Friedman Rule, Money Holdings Distributions

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*Research Institute for Economics & Business Administration (RIEB), Kobe University (E-mail: kkamiya@rieb.kobe-u.ac.jp)

[†]Graduate School of Economics and Management, Tohoku University (E-mail: so.kubota.c1@tohoku.ac.jp)

1 Introduction

Money is essential in decentralized exchange because agents must carry purchasing power into bilateral meetings. Without a centralized market to rebalance holdings, the terms of trade depend on how much money each party brings to a match. The distribution of money holdings should therefore matter for monetary policy, but canonical tractable models largely suppress this channel. In representative-agent models with a centralized market (Sidrauski, 1967; Lucas and Stokey, 1983), the distribution is degenerate, so the terms of trade do not matter. The same holds in models that pair a decentralized market with a recurring centralized one, which collapses money-holding heterogeneity each period (Lagos and Wright, 2005). Heterogeneous-agent models make non-degenerate distributions central to monetary transmission (Kaplan et al., 2018), but trade clears in Walrasian markets, so distributional effects operate mainly through consumption and saving responses rather than bilateral terms of trade. This paper studies monetary policy when exchange is decentralized and no centralized market intervenes, so that the endogenous distribution of money holdings shapes the terms of trade and, through them, the welfare effects of monetary policy. In particular, we ask whether this decentralized channel overturns the classical and widely accepted Friedman rule.

To address this question, we build on the search-and-bargaining tradition of Kiyotaki and Wright (1989, 1993) and the recent extension by Kamiya and Kubota (2025), which introduces divisible money and goods with linear production and a fixed cost. We develop a more tractable version of this framework by having each agent choose *ex ante* to be a buyer or a seller and by treating the fixed cost as sunk. Transactions are settled by Nash bargaining, and because buyers deplete their balances while sellers accumulate them, the distribution of money holdings is endogenously non-degenerate. We introduce the “pay-all equilibrium,” in which buyers exhaust their entire money holdings in every transaction. As a result, in the baseline economy positive money holders become buyers and zero-money holders become sellers, making each agent’s individual state transition straightforward. Because value functions are linear within each role, the value and aggregate equations reduce to finite-dimensional linear systems. This allows us to derive closed-form stationary pay-all equilibria and necessary and sufficient existence conditions.¹

¹Shi (1995) and Trejos and Wright (1995) study the case where only goods are divisible, while Green and Zhou (1998) and Zhou (1999) study the case where only money is divisible. In a general framework

Our main focus is on policy analysis. Friedman (1969) introduced the helicopter drop as a one-time thought experiment in which money is unexpectedly added to a community and collected by its members. We study two related policies. The first is a recurring drop governed by a constant money growth rule: every agent receives an equal lump-sum transfer each period, so the money stock grows at a constant rate $\mu \geq 0$. The second follows the timing of Friedman’s original experiment: every agent receives one unexpected equal transfer, after which the money stock remains constant.

We analyze the recurring policy first. Although the transfer is uniform in nominal terms, it is inherently redistributive whenever money holdings are heterogeneous: the implicit inflation tax falls on money-rich buyers, while the equal transfer is proportionally largest for money-poor sellers, so an equal drop shifts real purchasing power from buyers toward sellers.

Our analysis identifies two distinct channels through which monetary policy affects welfare, and their interplay determines the sign of the policy effect:

1. Extensive margin (seller mass): A recurring drop changes the money-holdings distribution and hence the seller mass.² This changes the mass of buyer–seller matches: when sellers are scarce, an increase in their number raises trade; when sellers are abundant, it reduces trade. A larger seller mass also raises total sunk costs.
2. Intensive margin (surplus per match and redistribution): Buyers and sellers value an additional unit of money differently. Sellers assign it a lower marginal value because they cannot spend it while selling and must wait until they become buyers in a future period. An equal drop therefore changes the surplus generated per match and has direct inflation-drift effects on welfare: it erodes buyers’ real balances while increasing sellers’ real balances.

Combining these channels yields our main policy result. Stationary welfare is strictly increasing in the money-growth rate over the whole range of positive rates for which the stationary equilibrium exists. This result challenges the optimality of the Friedman rule, though the

with divisible money, Kamiya and Shimizu (2006, 2007) show that equilibrium indeterminacy generically arises with discrete money-holdings distributions even when goods are divisible. Moreover, Kamiya and Shimizu (2011) presents a model in which indeterminacy also obtains under a continuum distribution of money holdings.

²Rocheteau and Wright (2005) define the extensive margin as the number of trades. In our framework, we define it in terms of the distribution of money holdings, since this distribution determines both the number of trades and the number of agents who incur the fixed cost.

uniform-transfer policy cannot implement negative money growth because agents with sufficiently low balances cannot surrender a common lump-sum tax.³

We next study the transition after a one-time drop. The transfer initially shifts real purchasing power toward sellers, and subsequent trade gradually redistributes money to buyers. The transition is governed by a three-dimensional linear system. Its bounded solution follows the system's single stable root and converges geometrically to the original real allocation with a proportionally larger nominal money stock. Under explicit conditions on primitives, output and welfare rise during the transition.

The transition mimics New Keynesian price dynamics through a different mechanism. Output responds on impact and then decays geometrically, while the price level adjusts only gradually to its new long-run value; depending on the initial equilibrium, the impact response of prices can be inflationary or deflationary, and it fades out thereafter. Gradual price adjustment alongside a real effect of money is the New Keynesian hallmark, yet here prices are perfectly flexible: every match is priced by Nash bargaining, with no price-setting friction. The sluggishness is instead distributional, because the money-holdings distribution adjusts slowly as money is redistributed through trade. Non-neutrality arises from search frictions and the slow diffusion of money, not from nominal rigidities.⁴

We next review the literature on policy effects operating through money-holdings distributions. Our model can be viewed as a divisible-money version of Trejos and Wright (1995). In a pure search environment, Molico (2006) studies these effects numerically. When a centralized market is added, as in Lagos and Wright (2005), the distributional consequences of inflation and monetary policy have been numerically examined by Chiu and Molico (2010, 2011), and more recently, by Bustamante (2023). Although not exactly a Lagos-Wright framework, Williamson (2006) develops a related alternating CM-DM model and shows how the equilibrium money distribution alters the optimality of the Friedman rule.

Further related contributions include Rocheteau et al. (2018), who develop a tractable model of monetary exchange with ex post heterogeneity, and Rocheteau et al. (2021), who

³Appendix A studies a transfer targeted to buyers, which admits negative growth. Within the construction studied there, welfare is again strictly increasing in the growth rate.

⁴Gradual adjustment of the price level to a money injection also arises in segmented-markets models (Alvarez et al., 2002), in which only a subset of agents absorbs new money on impact. Our mechanism is related in spirit, in that the injection reaches active trades only gradually, but it operates through search frictions and the endogenous money-holdings distribution rather than through limited participation in asset markets.

construct a heterogeneous-agent New Monetarist model with an application to unemployment. These studies demonstrate how the distribution of money holdings shapes welfare under alternative policies, combining analytic tractability with quantitative analysis.⁵ Our model differs from these contributions in that it requires no centralized market: the terms of trade in every match are set by Nash bargaining and depend on the balances of both parties. Combined with linear utility and costs, this structure yields a global stationary welfare comparison and an exact on-path transition after a one-time injection.

Recent research in heterogeneous-agent New Keynesian (HANK) models (e.g., Kaplan et al. (2018), Auclert (2019), and Auclert et al. (2025)) emphasize redistribution channels of monetary policy similar to ours: changes in interest rates and inflation redistribute wealth across agents with different marginal propensities to consume, thereby affecting aggregate demand. In these frameworks, the interaction between distribution and policy is typically assessed through numerical simulations. Our model shares this emphasis on redistribution but differs in two respects. First, whereas HANK models are built on Walrasian markets with nominal rigidities, all transactions in our model proceed through a search-and-bargaining market, so that the terms of trade are determined by Nash bargaining rather than by price-taking behavior. The distribution of money holdings thereby shapes the outside options of buyers and sellers in every bilateral match, creating a direct link between the wealth distribution and allocative efficiency that operates through the heterogeneity of the marginal value of money rather than through MPC heterogeneity. Second, the analytical tractability of our framework allows us to derive closed-form expressions for the steady-state welfare effects and for the on-path dynamics following a one-time helicopter drop, whereas HANK analyses typically study these interactions through quantitative methods.

Finally, we review the literature on dynamics in pure monetary search models. Trejos and Wright (1995) and Coles and Wright (1998) analyze search models with divisible goods and indivisible money, identifying various transition paths but without fully solving the dynamics. Green and Zhou (2002) is exceptional in analyzing dynamics with divisible goods and money without relying on numerical computations; although they do not obtain closed-form solutions, they prove the existence of equilibrium paths from any initial point to a

⁵Other related work includes Deviatov and Wallace (2001) on welfare-improving money creation in a Kiyotaki-Wright model; Jin and Zhu (2019) on money injections and distributions in a Trejos-Wright model; Williamson (2006) showing Friedman-rule inefficiency with alternating CM/DM participation; and Berentsen et al. (2005) on welfare-improving transfers with unequal allocations.

stationary equilibrium. Chiu and Molico (2021) investigate the short-run dynamics of a one-time money injection and show that it can have persistent positive effects on output, prices, and welfare, where heterogeneity in money holdings plays an important role, but they rely on numerical computation. Thus, although these papers richly characterize transitional dynamics, few deliver closed-form solutions.

The remainder of the paper is organized as follows. Section 2 introduces the economic environment. Section 3 describes the pay-all equilibrium and its key properties. Section 4 establishes equilibrium existence. Section 5 studies the helicopter drop to all agents under constant money growth and its welfare implications. Section 6 studies the one-time injection and solves its on-path dynamics in closed form. Section 7 concludes.

2 Economic environment

Time is discrete with an infinite horizon, $t = 1, 2, \dots$. There is a continuum of agents of measure one. Agents discount future payoffs by $\beta \in (0, 1)$. Each agent can produce a single divisible, non-storable good. Producing $x \geq 0$ units of the good imposes disutility $u^s(x) = -cx$, where $c \in (0, 1)$ is the unit cost. To eliminate the double-coincidence of wants, no agent can consume the good the agent produces, so an agent derives utility $u^b(x) = x$ from consuming another agent's output. There is also money, a perfectly divisible and storable medium of exchange. The total money supply is fixed at $M > 0$, and each agent's money holding m is a nonnegative real number.

The timeline in each period is as follows. At the beginning of period t , agents observe the economy-wide distribution of money holdings, denoted by a Borel probability measure H on $\mathbb{R}_+$. Each agent with money holding m then chooses either to be a buyer (B) or a seller (S) according to a Borel-measurable function $f: [0, \infty) \rightarrow \{B, S\}$. Agents who choose S pay the sunk production cost $d > 0$ prior to matching.⁶ Subsequently, each agent is randomly matched with another agent with probability $\alpha \in (0, 1)$, where the partner is drawn from the population distribution. In each match, both parties observe their partner's money holdings and role choice. Trade can occur only in a buyer–seller match; otherwise, no trade takes place.

In a buyer–seller match, the quantity of goods $x(m_b, m_s, H)$ and the monetary pay-

⁶Rocheteau and Wright (2005) also introduce an entry cost for becoming a seller.

ment $p(m_b, m_s, H)$ are determined by generalized Nash bargaining with the buyer's bargaining power $\theta \in (0, 1)$, where m_b and m_s denote the buyer's and the seller's money holdings, respectively.⁷ When no feasible agreement yields nonnegative surplus, the solution is $x(m_b, m_s, H) = 0$ and $p(m_b, m_s, H) = 0$. At the end of the period, money holdings update according to the realized payments, matches dissolve, and the economy proceeds to the next period.

Let H_B and H_S denote the money-holding distributions of buyers and sellers induced by H and f ; that is, for any Borel set $D \subset \mathbb{R}_+$, $H_B(D) = H(D \cap f^{-1}(B))$ and $H_S(D) = H(D \cap f^{-1}(S))$. Let $\bar{H}^S \equiv H_S(\mathbb{R}_+)$ be the measure of sellers, so the measure of buyers is $1 - \bar{H}^S$. Hence, each B agent meets an S agent with probability $\alpha \bar{H}^S$, and each S agent meets a B agent with probability $\alpha(1 - \bar{H}^S)$.

The stationary monetary equilibrium is defined as follows:

Definition 1. *Let v be a function from $\mathbb{R}_+$ to $\mathbb{R}$. A triple (H, f, v) is called a stationary monetary equilibrium if*

1. *the value function satisfies the Bellman equation*

$$\begin{aligned} v^B(m) &= \alpha \int_0^\infty [u^b(x(m, m_s, H)) + \beta v(m - p(m, m_s, H))] dH_S(m_s) \\ &\quad + (1 - \alpha \bar{H}^S) \beta v(m), \\ v^S(m) &= \alpha \int_0^\infty [u^s(x(m_b, m, H)) + \beta v(m + p(m_b, m, H))] dH_B(m_b) \\ &\quad + [1 - \alpha(1 - \bar{H}^S)] \beta v(m) - d, \\ v(m) &= \max\{v^B(m), v^S(m)\}. \end{aligned}$$

2. *H is stationary under the transition induced by matching, role choice f , and the payment rule p : for any Borel set $D \subset \mathbb{R}_+$, define $Q_B(m_b, D) = \{m_s \in \mathbb{R}_+ : m_b - p(m_b, m_s, H) \in D\}$ and $Q_S(m_s, D) = \{m_b \in \mathbb{R}_+ : m_s + p(m_b, m_s, H) \in D\}$. Then*

$$\begin{aligned} H(D) &= (1 - \alpha \bar{H}^S) H_B(D) + [1 - \alpha(1 - \bar{H}^S)] H_S(D) \\ &\quad + \alpha \int_0^\infty H_S(Q_B(m_b, D)) dH_B(m_b) + \alpha \int_0^\infty H_B(Q_S(m_s, D)) dH_S(m_s), \end{aligned}$$

⁷Replacing the Nash bargaining solution with Kalai's (1977) proportional solution yields qualitatively the same results; see Section 8.1 of Kamiya and Kubota (2025).

3. $x(m_b, m_s, H) \geq 0$ and $p(m_b, m_s, H) \geq 0$ solve each Nash bargaining problem:

- if there exists (x, p) such that $x > 0$, $0 \leq p \leq m_b$, and both the buyer's and seller's surpluses are nonnegative, i.e.,

$$x + \beta(v(m_b - p) - v(m_b)) \geq 0 \text{ and } -cx + \beta(v(m_s + p) - v(m_s)) \geq 0,$$

then the trade $(x(m_b, m_s, H), p(m_b, m_s, H))$ is determined by

$$\begin{aligned} (x(m_b, m_s, H), p(m_b, m_s, H)) = \\ \arg \max_{x,p} [x + \beta(v(m_b - p) - v(m_b))]^\theta [-cx + \beta(v(m_s + p) - v(m_s))]^{1-\theta} \\ \text{s.t. } x > 0, \quad 0 \leq p \leq m_b, \end{aligned}$$

- otherwise $x(m_b, m_s, H) = p(m_b, m_s, H) = 0$.

4. $v(m)$ is strictly increasing.

The last condition ensures that money has positive value in equilibrium.

For later use, define the *buyer's surplus* and the *seller's surplus* in a meeting between a buyer holding m_b and a seller holding m_s as

$$J^b(m_b, m_s, H) \equiv x(m_b, m_s, H) + \beta[v(m_b - p(m_b, m_s, H)) - v(m_b)], \quad (1)$$

$$J^s(m_b, m_s, H) \equiv -cx(m_b, m_s, H) + \beta[v(m_s + p(m_b, m_s, H)) - v(m_s)]. \quad (2)$$

3 Pay-all equilibrium

We focus on the “pay-all” equilibrium, a specific subclass of stationary monetary equilibria, and show that it is sufficiently tractable to establish existence and to obtain analytical characterizations of its properties and transitional dynamics.

Definition 2. A stationary monetary equilibrium is called a *pay-all equilibrium* if the following two conditions are satisfied. At each Nash bargaining,

1. if trade occurs, i.e., $x(m_b, m_s, H) > 0$, the monetary payment is binding:

$$p(m_b, m_s, H) = m_b;$$

2. there exists $\tilde{m} > 0$ such that $f(m) = S$ for $m \in [0, \tilde{m})$ and $f(m) = B$ otherwise.

We call this a pay-all equilibrium because, by condition (i), each buyer spends all money holdings upon agreement. Condition (ii) means that an agent chooses S if the agent's balances are not sufficiently large and chooses B otherwise. We refer to the two conditions in Definition 2 collectively as the *pay-all property*.

We will show that the pay-all property is driven by linear utility and linear variable costs together with search frictions: the marginal utility of an additional unit of consumption today exceeds the discounted continuation value, leaving buyers with no incentive to save money. Hence, in equilibrium buyers fully deplete balances upon meeting sellers.

Let $H(0) \in (0, 1)$ denote the mass of agents holding no money. Under the pay-all property, a matched buyer spends the entire balance and carries zero money into the next period, whereas a matched seller receives the buyer's balance and thereby enters the buyer region. Throughout, a superscript $*$ denotes the value of an object in a stationary equilibrium. As we show in Section 4, stationarity together with the pay-all property pins down the following distributional restrictions: $H(0) = H^*(0)$ and $H((0, \tilde{m})) = 0$ together with

$$\int_{\tilde{m}}^{\infty} dH(m) = 1 - H^*(0), \quad (3)$$

$$\int_{\tilde{m}}^{\infty} m dH(m) = M. \quad (4)$$

where $H^*(0)$ is the equilibrium mass of zero-money holders determined in Lemma 1 below. The condition $H((0, \tilde{m})) = 0$ rules out balances in $(0, \tilde{m})$: no agent ever enters this interval because matched buyers jump to zero while matched sellers jump to $[\tilde{m}, \infty)$. Hence the remaining mass $1 - H^*(0)$ lies entirely on $[\tilde{m}, \infty)$ and consists of buyers, whose average balance is $M/(1 - H^*(0))$.

The law of motion of the cross-sectional distribution is correspondingly simple. In each period, a mass $\alpha H^*(0)(1 - H^*(0))$ switches roles: matched buyers exhaust their balances and become zero-money holders, while matched sellers receive those balances and become buyers. In a stationary equilibrium these offsetting flows are equal, which keeps the aggregate dynamics tractable. This contrasts with Lagos and Wright (2005), where the centralized market reallocates balances each period and collapses the distribution to a single point.

Although (3) and (4) pin down the total mass and total money held by buyers, they do

not pin down the shape of the buyer distribution on $[\tilde{m}, \infty)$. A continuum of distributions is consistent with the pay-all equilibrium, giving rise to a micro-level indeterminacy analogous to that documented in Kamiya and Kubota (2025). As shown below, however, this indeterminacy is irrelevant for the aggregates of interest: output, welfare, and the average joint surplus are uniquely determined by parameters.

In what follows, we focus on on-path trades. Specifically, we consider meetings with $(m_b, m_s) = (m, 0)$ and write $x(m) \equiv x(m, 0, H)$, $p(m) \equiv p(m, 0, H)$, $J^b(m) \equiv J^b(m, 0, H)$, and $J^s(m) \equiv J^s(m, 0, H)$ for $m \geq 0$. Given the corner solution $p(m) = m$, Nash bargaining determines the quantity by

$$x(m) = \arg \max_x \left[x + \beta(v(0) - v(m)) \right]^\theta \left[-cx + \beta(v(m) - v(0)) \right]^{1-\theta}, \quad (5)$$

with the first-order condition

$$(1 - \theta)c \left[x + \beta(v(0) - v(m)) \right] = \theta \left[-cx + \beta(v(m) - v(0)) \right]. \quad (6)$$

The second-order condition holds globally since the logarithm of the Nash product is concave. Equation (6) can be rewritten as

$$cx(m) = \beta[\theta + (1 - \theta)c](v(m) - v(0)). \quad (7)$$

Joint surplus and effective bargaining power. Define the *joint surplus* in bargaining as

$$J(m) \equiv (1 - c)x(m). \quad (8)$$

Since the buyer's marginal utility is 1 and the seller's marginal cost is c , each unit of x generates a net gain of $1 - c$, so $J(m)$ measures the total gains from trade. From (6), the Nash bargaining solution splits $J(m)$ with weights summing to 1:

$$J^b(m) = \tilde{\theta} J(m), \quad (9)$$

$$J^s(m) = (1 - \tilde{\theta}) J(m), \quad (10)$$

where the *effective bargaining power* is

$$\tilde{\theta} \equiv \frac{\theta}{\theta + (1 - \theta)c}, \quad 1 - \tilde{\theta} = \frac{(1 - \theta)c}{\theta + (1 - \theta)c}. \quad (11)$$

Note that $\tilde{\theta} > \theta$ whenever $c < 1$. Since compensating the seller is cheap (giving one more unit of x costs the seller only c but benefits the buyer by 1), Nash bargaining exploits this asymmetry, tilting the surplus split toward the buyer. This cost-sensitivity of the split is also the economic reason we adopt Nash bargaining as the trading protocol: when production becomes cheaper, the division of the gains from trade should respond to how cheaply the seller can be compensated with output, and the Nash solution internalizes exactly this margin, whereas a proportional split leaves the division unchanged and lets only the total surplus adjust.⁸

On the equilibrium path ($m \in \{0\} \cup [\tilde{m}, \infty)$), the value functions satisfy

$$\begin{aligned} v(m) &= \alpha H(0) [x(m) + \beta v(0)] + (1 - \alpha H(0)) \beta v(m) \\ &= \alpha H(0) \tilde{\theta} J(m) + \beta v(m), \quad \text{for } m \in [\tilde{m}, \infty), \end{aligned} \quad (12)$$

$$\begin{aligned} v(0) &= \alpha \int_{\tilde{m}}^{\infty} [-cx(m) + \beta v(m)] dH(m) + [1 - \alpha(1 - H(0))] \beta v(0) - d \\ &= -d + \alpha \int_{\tilde{m}}^{\infty} (1 - \tilde{\theta}) J(m) dH(m) + \beta v(0). \end{aligned} \quad (13)$$

In each equation, the first equality is the standard Bellman recursion, and the second equality uses the surplus split (9)–(10) to express the match payoff in terms of $J(m)$.

Intuitively, agents with small money holdings $m < \tilde{m}$ choose S because the money they could offer is too little to induce a seller to produce; they are better off choosing S and accumulating money through selling. The full Bellman equation system, including off-path money holdings $m \notin 0 \cup [\tilde{m}, \infty)$, is given in the next section.

Measure of zero-money holders and the value at zero. We now determine $v(0)$ and the measure of potential sellers $H(0)$ in pay-all equilibria.

⁸If Nash bargaining is replaced by Kalai's proportional solution, the surplus is split proportionally to the bargaining weights, so that $J^b(m)/\theta = J^s(m)/(1 - \theta)$. Combined with $J(m) = J^b(m) + J^s(m)$, this directly yields $J^b(m) = \theta J(m)$. Thus, the effective bargaining power under Kalai bargaining is exactly $\tilde{\theta} = \theta$, making the surplus split independent of the production cost c .

Lemma 1. *In any pay-all equilibrium,*

$$H^*(0) = \frac{(1 - \beta)c}{\alpha\beta\theta(1 - c)}, \quad (14)$$

$$v^*(0) = 0. \quad (15)$$

Proof. From (7) and (8), the joint surplus can be expressed via the Nash FOC as

$$J(m) = \frac{(1 - c)\theta}{c\tilde{\theta}} \beta(v(m) - v(0)). \quad (16)$$

From (12), the buyer's Bellman equation gives

$$J(m) = \frac{1 - \beta}{\alpha\tilde{\theta}H(0)} v(m). \quad (17)$$

For both (16) and (17) to hold for all $m \geq \tilde{m}$ in a stationary monetary equilibrium, where $v(m)$ is strictly increasing, we must have $v(0) = 0$ and the coefficients of $v(m)$ on the right-hand sides must coincide. Equating them yields (14). $\square$

In the proof, equating two independent expressions for $J(m)$, one from the intratemporal surplus split (Nash bargaining) and the other from the intertemporal value recursion (Bellman equation), yields (14) and (15). Lemma 1 implies $H^*(0) \in (0, 1)$ whenever

$$\alpha\beta\theta(1 - c) > (1 - \beta)c.$$

Economically, higher α or θ , or lower c , makes each buyer-seller meeting more lucrative: higher α means trading partners are easier to find, higher θ means buyers capture a larger share of the surplus, and lower c widens the per-unit gain from trade, $1 - c$. As the expected return from being a buyer rises relative to the sunk cost d of being a seller, more agents find it worthwhile to hold money and become buyers, shrinking the seller pool $H^*(0)$ and enlarging the buyer pool. Conversely, a lower β compresses continuation values: future trade gains are discounted more heavily, weakening the incentive to hold money across periods, so more agents remain in the zero-money (seller) state and $H^*(0)$ rises.

In Lagos and Wright (2005), the centralized market disciplines price and quantity choices each period. By contrast, in our environment without a centralized market, the nominal price

level is not pinned down by Walrasian clearing, whereas the real allocation on the equilibrium path is disciplined by the pay-all property and the endogenously determined measure $H^*(0)$.

Macro-level variables. Given $H^*(0)$ and $v^*(0) = 0$, macro-level variables can be solved sequentially. Define the average continuation value of positive-balance agents:

$$\bar{v}_b \equiv \frac{\int_{\tilde{m}}^{\infty} v(m) dH(m)}{1 - H^*(0)}. \quad (18)$$

Total output is

$$Y \equiv \alpha H^*(0) \int_{\tilde{m}}^{\infty} x(m) dH(m), \quad (19)$$

where H is any probability distribution satisfying (3)–(4) and $H((0, \tilde{m})) = 0$. Here, $x(m)$ is the production when a buyer with m meets a zero-money seller; if all buyers matched, total production would be $\int_{\tilde{m}}^{\infty} x(m) dH(m)$, which we scale by the meeting probability $\alpha H^*(0)$ to obtain actual production. As will be shown later, Y is uniquely determined irrespective of the choice of H satisfying these distributional restrictions. The average production per successful meeting is therefore

$$\bar{x} \equiv \frac{Y}{\alpha H^*(0)(1 - H^*(0))} = \frac{\int_{\tilde{m}}^{\infty} x(m) dH(m)}{1 - H^*(0)}. \quad (20)$$

Define the average joint surplus as $\bar{J} \equiv (1 - c) \bar{x}$. Integrating the Bellman equations (12)–(13) over $m \in [\tilde{m}, \infty)$ and using $v^*(0) = 0$ yields

$$(1 - \beta) \bar{v}_b = \alpha H^*(0) \tilde{\theta} \bar{J}, \quad (21)$$

$$d = \alpha(1 - H^*(0))(1 - \tilde{\theta}) \bar{J}. \quad (22)$$

The buyer's discounted value (21) equals the expected share $\tilde{\theta} \bar{J}$ of the joint surplus. The seller's sunk cost (22) is covered by the expected share $(1 - \tilde{\theta}) \bar{J}$.

Proposition 1 (Aggregate uniqueness). *Assume $(1 - \beta)c < \alpha\beta\theta(1 - c)$. Every pay-all*

equilibrium has the seller mass $H^*(0)$ given by (14) and the same aggregate outcomes:

$$\bar{J}^* = \frac{d}{\alpha(1 - H^*(0))(1 - \tilde{\theta})}, \quad \bar{v}_b^* = \frac{\alpha H^*(0) \tilde{\theta}}{1 - \beta} \bar{J}^* = \frac{d}{Z}, \quad \bar{x}^* = \frac{\bar{J}^*}{1 - c}, \quad (23)$$

where $Z \equiv \alpha\beta(1 - c)(1 - \theta)(1 - H^*(0))$.

Proof. Lemma 1 determines $H^*(0)$. Equation (22) then determines $\bar{J}^*$; substituting it into (21) determines $\bar{v}_b^*$, and $\bar{x}^* = \bar{J}^*/(1 - c)$ follows from the definition of joint surplus. $\square$

Thus, although the macro variables $\bar{J}^*$, $\bar{v}_b^*$, and $\bar{x}^*$ are uniquely determined by parameters, the equilibrium is indeterminate at the micro level in two independent dimensions. First, the distributional restrictions (3)–(4) pin down the total mass and total money held by buyers but not the shape of H on $[\tilde{m}, \infty)$, so a continuum of buyer distributions is consistent with equilibrium. Second, the value-function profile is free in two dimensions: the cutoff $\tilde{m}$ may sit anywhere in $(0, M/(1 - H^*(0)))$, and the value function may carry an upward jump $\Delta^v \geq 0$ at the cutoff; for each admissible pair, the construction of Section 4 pins the two slopes, and hence the whole profile, in closed form (Lemma 2 below), the profile being disciplined only by no-deviation inequalities. These sources of indeterminacy, the profile and the distribution, are independent: any admissible H can be paired with any admissible profile. Because Nash bargaining links individual production $x(m)$ to $v(m)$ through (7), the value-function indeterminacy is real rather than nominal: buyers at the same balance trade different quantities across equilibria, even though aggregate output and welfare are invariant. This micro-level indeterminacy disappears under the constant-drop policy: Proposition 3 selects a unique stationary equilibrium.

Define social welfare as

$$W \equiv H^*(0) v^*(0) + \int_{\tilde{m}}^{\infty} v(m) dH(m) = (1 - H^*(0)) \bar{v}_b^* = \frac{d}{\alpha\beta(1 - c)(1 - \theta)}. \quad (24)$$

A counter-intuitive implication is that a higher sunk cost d raises average output $\bar{x}^*$, the average buyer value $\bar{v}_b^*$, and welfare W . The key is that d is sunk: it does not enter $J^s(m_b, m_s, H)$ inside the Nash product and therefore does not distort the terms of trade directly. Instead, it enters only through the seller's Bellman equation (22),

$$d = \alpha(1 - H^*(0))(1 - \tilde{\theta}) \bar{J}.$$

Therefore, when d rises, the equilibrium must generate a larger expected gain from trade in order to compensate sellers for becoming active. This requires a higher average joint surplus $\bar{J}$ and hence a higher average output $\bar{x}$. The increase in $\bar{J}$ then raises $\bar{v}_b^*$ and welfare. In this sense, the role of d is analogous to a free-entry condition in labor-search models: sellers incur the sunk cost to enter the market, and in equilibrium the expected gains from trade adjust to cover that cost. We stress that this comparative static leans on the linear production technology and the absence of a capacity limit (the sunk cost adjusts participation while output per match is free to expand), so we read it as a property of this equilibrium rather than as a general policy implication; with bounded output or convex costs it could reverse.

4 Existence

We establish the existence of pay-all equilibria by a guess-and-verify approach: we construct a candidate equilibrium and then verify that it satisfies all equilibrium conditions.

In any pay-all equilibrium, the value function v must satisfy the following Bellman equations at every money holding $m \in [0, \infty)$, on and off the equilibrium path:

$$v(m) = \alpha H(0)(x(m, 0, H) + \beta v(0)) + (1 - \alpha H(0))\beta v(m) \text{ for } m \in [\tilde{m}, \infty), \quad (25)$$

$$v(m) = \alpha \int_{\tilde{m}}^{\infty} (-cx(m_b, m, H) + \beta v(m + m_b)) dH(m_b) + [1 - \alpha(1 - H(0))]\beta v(m) - d$$

for $m \in [0, \tilde{m})$. (26)

The tractability of the pay-all equilibrium comes from a simple implication of buyers exhausting their balances. When a buyer with $m \geq \tilde{m}$ meets a seller with $m_s = 0$, the buyer spends all the money. As a result, the on-path value recursion is linear in m , and the cross-sectional dynamics reduce to the single aggregate $H(0)$. Existence of a pay-all equilibrium therefore requires that the per-meeting surplus be large enough on two fronts: it must make spending the entire balance optimal for the buyer, and the resulting bargaining share must, in expectation, cover the seller's sunk cost d . Proposition 2 below shows that both requirements reduce to a single restriction on parameters.

The construction restricts the value function to a tractable form.

Definition 3 (two-branch linear pay-all equilibrium). *A pay-all equilibrium is two-branch*

linear if

1. for some anchor v_0 , cutoff $\tilde{m} > 0$, slopes $b > a > 0$, and jump $\Delta^v \geq 0$, its value function is

$$v(m) = \begin{cases} v_0 + am, & m \in [0, \tilde{m}), \\ v_0 + a\tilde{m} + \Delta^v + b(m - \tilde{m}), & m \in [\tilde{m}, \infty); \end{cases} \quad (27)$$

2. an agent moves between the two branches of v only through trade.

In the baseline economy, Clause 2 holds automatically because balances change only through trade. It becomes restrictive once monetary transfers are introduced in Section 5.

Clause 1 allows the two branches to be disconnected, so continuity at the cutoff is not imposed; $\Delta^v \geq 0$ keeps v increasing. The pay-all property makes the trade surplus and the continuation value affine in m on each branch, but constant slopes across the whole of each role region are a restriction on the equilibria we study, not an implication. The anchor is kept general because the baseline derives $v_0 = 0$ (Lemma 1), whereas the equilibrium under the recurring drop has $v_0 > 0$.

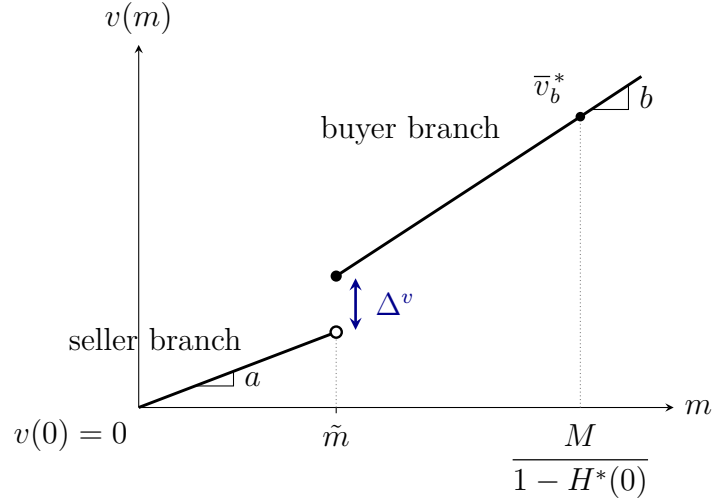
The cutoff rule itself is already part of Definition 2, which also fixes the roles that can trade: an S -agent with $m_s < \tilde{m}$ and a B -agent with $m_b \geq \tilde{m}$. That such a meeting does trade is not imposed here; the proof of Proposition 2 verifies a strictly positive surplus for the profiles of Definition 3 under the pay-all bound on Δ^v . Meetings of two S -agents or two B -agents contain no producer–consumer pair.

The cutoff $\tilde{m}$ and the jump Δ^v are free parameters of the construction; given the pair, the slopes, and hence the whole profile, are uniquely pinned in closed form, as the following lemma shows. The equilibrium is thus indeterminate at the micro level, in the cutoff, in the jump, and in the distribution, even though the aggregate objects remain uniquely determined.

Lemma 2 (The value-function family). *Fix a cutoff $\tilde{m} \in (0, M/(1 - H^*(0)))$ and a jump $\Delta^v \in [0, \bar{\Delta}(\tilde{m})]$, with $\bar{\Delta}$ the pay-all bound (B.20) of the appendix. In any two-branch linear pay-all equilibrium,*

$$a = \chi b, \quad \chi = \frac{s}{1 - \beta + s} \in (0, 1), \quad s \equiv \alpha\beta(1 - \theta)(1 - H^*(0)), \quad (28)$$

Figure 1: Equilibrium value function



Note: In the baseline, $v(0) = 0$. The seller branch has slope a ; the buyer branch has slope $b > a$, starts $\Delta^v \geq 0$ above the seller branch at the cutoff, and passes through the average buyer's pair $(M/(1 - H^*(0)), \bar{v}_b^*)$. Because $a = \chi b$ with $\chi \in (0, 1)$, any change in b is accompanied by a proportional change in a . The continuous member $\Delta^v = 0$ is the boundary case of the family.

and

$$b = b(\tilde{m}, \Delta^v) \equiv \frac{d/Z - \Delta^v}{M/(1 - H^*(0)) - (1 - \chi)\tilde{m}} > 0, \quad (29)$$

Proof. See Appendix B.1.1. □

Here s is the discounted per-period rate at which a seller's marginal balance is revalued to the buyer slope b : a seller meets a buyer with probability $\alpha(1 - H^*(0))$ and keeps the share $1 - \theta$ of the marginal surplus, so $\chi < 1$ reflects the seller's waiting time until trade. Equation (29) is equivalent to $v(M/(1 - H^*(0))) = d/Z = \bar{v}_b^*$: every member of the family, whatever its jump, passes through the average buyer's balance–value pair pinned down by Proposition 1. A larger jump changes b and hence also $a = \chi b$, while the buyer branch continues to pass through this pivot; Figure 1 illustrates one representative member.

The following proposition shows that, under a single parameter restriction, a two-branch linear pay-all equilibrium exists for every admissible cutoff and jump.

Proposition 2 (Existence). *Suppose*

$$\alpha > \frac{c(1 - \beta)}{\beta\theta(1 - c)}. \quad (30)$$

Then, for each cutoff $\tilde{m} \in (0, M/(1 - H^(0)))$ and each jump $\Delta^v \in [0, \bar{\Delta}(\tilde{m})]$, there exists a two-branch linear pay-all equilibrium, with the value function (27) and the slopes given in Lemma 2.*

Proof. See Appendix B.1.2. □

Condition (30) is equivalent to $H^*(0) < 1$, the requirement that a strictly positive measure of buyers be sustained in steady state. Because Lemma 1 determines $H^*(0)$ from parameters alone, the inequality is also necessary within the class of pay-all equilibria constructed above: when it fails, the formula yields $H^*(0) \geq 1$, leaving no room for buyers and ruling out trade. Intuitively, the per-meeting surplus, scaled by the matching probability α , the buyer's share θ , and the net gain $1 - c$, and discounted by β , must be large enough to give sellers the incentive to accept money in exchange for their production.

5 Constant Money Growth: A Helicopter Drop to All Agents

In Sections 5 and 6, we introduce monetary expansion to the model and analyze its welfare consequences. Because the money supply varies over time, the distinction between nominal and real balances is essential. Hereafter, nominal variables carry the superscript n (e.g., m_t^n for individual money holdings at period t), and real variables carry no superscript (e.g., m_t for the real balance).

We first formulate the common framework in a general non-stationary setting, and then move on to the stationary equilibrium. At the beginning of each period t , the government makes a transfer to agents. Let $\tau_t^n(m^n)$ denote the nominal period- t transfer received by an agent whose nominal balance at the end of period $t - 1$ is m^n . Let M_t^n denote the aggregate money supply in period t , measured immediately after the period- t transfer has been paid and before agents choose roles, match, and bargain. Similarly, let H_t^n denote the corresponding cross-sectional distribution of money holdings at the same instant. Suppose

the money growth rate is μ_t , so that $M_t^n = (1 + \mu_t)M_{t-1}^n$. The transfer rule must aggregate to the increase in the money supply:

$$\int_0^\infty \tau_t^n(m^n) dH_{t-1}^n(m^n) = \mu_t M_{t-1}^n, \quad (31)$$

where the integration is taken against H_{t-1}^n because pre-transfer balances at the start of period t coincide with balances at the end of period $t - 1$.

This section focuses on the case in which the money growth rate is constant at μ . The real balance is defined as $m_t \equiv m_t^n / (1 + \mu)^t$. In stationary equilibrium, the real money supply $M_t^n / (1 + \mu)^t$ is constant in t , and we denote this constant by M , so that $M = M_t^n / (1 + \mu)^t$ for all t .

5.1 Policy and Stationary Distribution

The policy. We consider the lump-sum transfer (helicopter drop) to all agents: $\tau_t^n(m^n) = \mu M_{t-1}^n$ for all $m^n \geq 0$ and $t = 1, 2, \dots$. Every agent, including those with zero money holdings, receives the transfer. The recurring-drop results below assume $\mu > 0$; negative growth is infeasible under a uniform transfer, since agents holding no money cannot pay a lump-sum tax, and we use $\mu = 0$ only as the limiting no-growth benchmark. Because the transfer also reaches agents below the role cutoff, sellers carry positive balances. As a result, the incidence of the transfer matters: the same unit of money has different continuation values depending on whether a seller or a buyer holds it, and this difference feeds into the bargaining outcomes.

Money dynamics and cutoff. We work with real balances $m \equiv m_t^n / (1 + \mu)^t$, and let $M \equiv M_t^n / (1 + \mu)^t$ denote the constant real money stock.

Under pay-all, an agent's balance follows a simple cycle. A buyer who trades pays the entire balance and ends the period with zero money. At the beginning of the next period, the transfer gives the agent a positive real balance below the cutoff M , so the agent becomes a seller. As long as the agent does not trade, successive transfers gradually increase the real balance toward M from below. When the agent meets a buyer, the agent produces the good and receives the buyer's entire balance, moving into the buyer region. Thereafter, as long as the agent does not trade, inflation gradually reduces the real balance toward M from above. When the agent next meets a seller, the agent pays the entire balance, returns to zero, and

the cycle begins again.

Formally, between trades the combination of the transfer and inflation maps the current real balance into

$$m_{t+1} = g(m_t) \equiv \frac{m_t + \mu M}{1 + \mu}.$$

The map g has the unique fixed point M and is a contraction when $\mu > 0$: it raises balances below M , lowers balances above M , and never carries a balance across M . Clause 2 of Definition 3 therefore holds if and only if $\tilde{m} = M$, and the recurring drop pins the cutoff that the baseline left free: sellers approach the cutoff from below, buyers approach it from above, and agents move between branches only when they trade. The clause is a restriction, not an implication of stationarity. With $\tilde{m} \neq M$ an unmatched agent's balance would drift across the cutoff and hence move to the other branch; the stationary flows would have to account for such transitions, which we do not study.

Because every agent receives the transfer, the drop moves balances deterministically between trades, which pins the distribution below M down to a grid and makes the stationary distribution unique within this class, as illustrated in Figure 2. This contrasts with the no-policy baseline, where the distribution over $[\tilde{m}, \infty)$ is indeterminate, as noted in the remark after Proposition 1. We characterize the two sides of the cutoff in turn, writing $L = [0, M)$ and $H^*(L) = \int_0^M dH^*(m)$ for the seller region and its mass.

Lemma 3 (Stationary distribution). *Suppose $\mu > 0$. In any stationary pay-all equilibrium under the policy of this section whose role regions satisfy clause 2 of Definition 3, equivalently with $\tilde{m} = M$:*

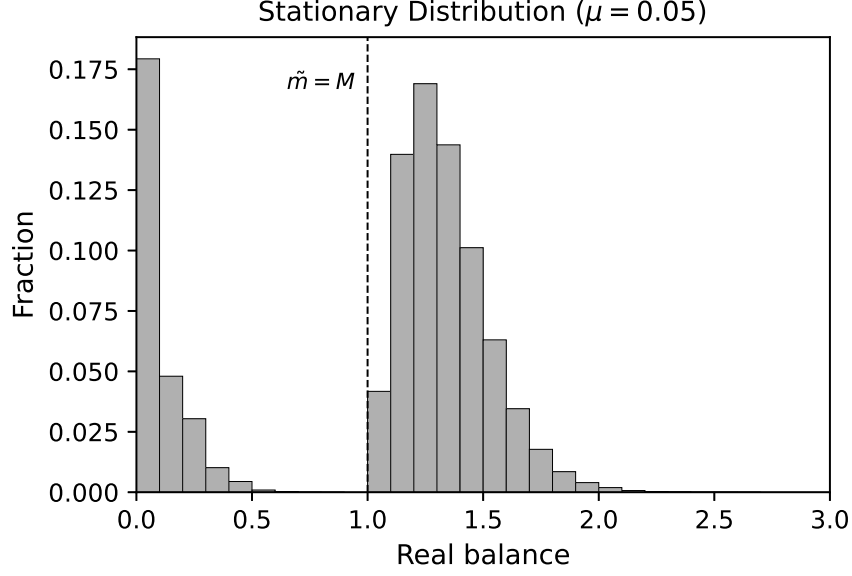
(i) (Seller side.) *The stationary distribution on L is supported on the grid*

$$\ell_k = M(1 - (1 + \mu)^{-k}), \quad k = 1, 2, \dots,$$

with masses

$$\omega_k \equiv H^*(\{\ell_k\}) = \alpha H^*(L)(1 - H^*(L)) \rho^{k-1}, \quad \rho \equiv 1 - \alpha(1 - H^*(L)),$$

Figure 2: Stationary distribution of real money holdings under constant money growth.



Note: Simulated with parameters $\alpha = 0.3$, $\beta = 0.95$, $c = 0.5$, $\theta = 0.9$, $\mu = 0.05$, $M = 1$. The dashed vertical line marks the buyer–seller cutoff $\tilde{m} = M$. Sellers ($m < M$) accumulate positive balances on a discrete grid driven by the per-period helicopter drop, in contrast to the baseline where all sellers are at $m = 0$; their masses decay geometrically along the grid, so most of the seller mass sits near zero. Buyers ($m \geq M$) have a unique stationary distribution on a bounded support, forming the right-hand bell of the histogram.

so that the stationary real amount of money held by agents in L is

$$M^* = \frac{H^*(L)\mu M}{\mu + \alpha(1 - H^*(L))}. \quad (32)$$

(ii) (Buyer side.) Let $B^*(y) \equiv H^*([M, y])$, with $B^*(y) = 0$ for $y < M$ and $B^*(y) = 1 - H^*(L)$ for $y \geq \bar{m}_{\max} \equiv (1 + \mu)M/\mu$. The buyer distribution has bounded support contained in $[M, \bar{m}_{\max}]$, and B^* is the unique solution of

$$\begin{aligned} B^*(y) &= (1 - \alpha H^*(L)) B^*((1 + \mu)y - \mu M) \\ &\quad + \alpha \int_0^M B^*((1 + \mu)y - \mu M - s) dH^*(s), \quad M \leq y \leq \bar{m}_{\max}. \end{aligned} \quad (33)$$

Proof. See Appendix B.2.1. □

The mechanics behind the lemma are simple. Write $H \equiv H^*(L)$ for short. A buyer who trades ends the period with zero money and enters the seller grid next period at $\ell_1 = g(0) = \mu M / (1 + \mu)$. A seller who does not trade moves deterministically from ℓ_k to ℓ_{k+1} , while leaving the seller region with probability $\alpha(1 - H)$. Stationarity therefore implies

$$\omega_1 = \alpha H(1 - H), \quad \omega_{k+1} = \rho \omega_k, \quad \rho = 1 - \alpha(1 - H).$$

Iterating this recursion gives the geometric masses in part (i). Summing $\sum_{k \geq 1} \omega_k \ell_k$ then gives (32).

For the buyer side, use the excess balance $Y = m - M$ and let T be the one-period transition operator on buyer distributions. The stationarity condition $T(\Phi) = \Phi$, written in terms of the CDF, is precisely (33), and the interval $[M, (1 + \mu)M/\mu]$ is invariant. Couple two buyer populations through the same trade arrivals and seller draws. Every balance gap is divided by $1 + \mu$ in one period, so T contracts the Wasserstein distance by $1/(1 + \mu)$. The stationary buyer distribution is therefore unique.

For the buyer side, inflation itself pins the distribution down. Between trades it shrinks any difference between two balances by the factor $1 + \mu$ each period, so two buyer populations facing the same trading shocks are drawn together over time, and only one stationary buyer distribution survives.

The lemma, together with the money-market identity $M = M^* + (1 - H^*(L)) \bar{m}_b$, gives the conditional mean balances

$$\bar{m}_s = \frac{M^*}{H^*(L)} = \frac{\mu M}{\mu + \alpha(1 - H^*(L))}, \quad \bar{m}_b = \frac{M - M^*}{1 - H^*(L)} = \frac{(\mu + \alpha)M}{\mu + \alpha(1 - H^*(L))}. \quad (34)$$

When $\mu = 0$, $M^* = 0$ and $\bar{m}_s = 0$: the seller side carries no money, recovering the baseline.

5.2 Equilibrium and Existence

Value function, joint surplus, and Bellman equations. As in the no-policy baseline of Section 4, we seek a two-branch linear pay-all equilibrium in the sense of Definition 3,

now with the cutoff at $\tilde{m} = M$:

$$v(m) = \begin{cases} v(0) + a m, & m \in [0, M), \\ v(0) + aM + \Delta^v + b(m - M), & m \in [M, \infty), \end{cases} \quad (35)$$

where a and b are the marginal values of money in the seller and buyer regions. The discontinuity at the cutoff is

$$\Delta^v \equiv v(M) - \lim_{m \uparrow M} v(m), \quad (36)$$

the value gained on crossing M . Later, we will show that $\Delta^v = 0$ and the uniqueness of equilibrium.

On the equilibrium path, a buyer with $m_b \geq M$ meets a seller with $m_s \in [0, M)$. As in Section 3, the Nash bargaining solution splits the joint surplus with effective bargaining power $\tilde{\theta}$. The new feature relative to the baseline is that the seller's pre-existing balance m_s is now positive, which adds a wedge term to the joint surplus. Under the value function (35), the joint surplus takes the explicit form

$$J(m_b, m_s) = (1 - c)x(m_b, m_s) + \frac{\beta(b - a)}{1 + \mu} m_s. \quad (37)$$

The first term is the production surplus. The second term arises because the buyer's payment m_b passes through the seller's low-valuation state (a) before reaching the buyer region (b): each unit of the seller's pre-existing balance m_s bundled with the payment generates an additional surplus of $\beta(b - a)/(1 + \mu)$. When sellers hold zero money, this term vanishes and J reduces to $(1 - c)x$. See Appendix B.2.

The Bellman equations take the same form as the surplus-split form in earlier sections. For a buyer with $m \geq M$,

$$v(m) = \alpha \tilde{\theta} \int_0^M J(m, m_s) dH^*(m_s) + \beta v\left(\frac{m + \mu M}{1 + \mu}\right). \quad (38)$$

For a seller with $m < M$,

$$v(m) = -d + \alpha(1 - \tilde{\theta}) \int_M^\infty J(m_b, m) dH^*(m_b) + \beta v\left(\frac{m + \mu M}{1 + \mu}\right). \quad (39)$$

These are direct analogues of (12)–(13), with two differences: (i) the surplus $J(m_b, m_s)$ depends on the seller's balance m_s , and (ii) the continuation values include an inflation drift because the real balance drifts toward M between trades.

Seller mass and slopes.

Lemma 4. *Suppose $\mu > 0$. In any stationary pay-all equilibrium of the form (35) under the policy $\tau_t^n(m^n) = \mu M_{t-1}^n$ for all $m^n \geq 0$,*

$$H^*(L) = \frac{c(1 - \beta + \mu)}{\alpha\beta\theta(1 - c)}, \quad (40)$$

$$a^* = \chi b^*, \quad \chi \equiv \frac{\alpha\beta(1 - H^*(L))(1 - \theta)}{1 - \beta + \mu + \alpha\beta(1 - H^*(L))(1 - \theta)}. \quad (41)$$

Proof. See Appendix B.2.2. □

No jump survives the drop. Whether a discontinuity can survive is settled by the role choice near the cutoff. A downward jump fails at the cutoff itself: exactly at M the seller role would pay strictly more than the assigned buyer role. An upward jump fails just below it: a seller close enough to M prefers to switch roles, pay its entire balance, push any partner across the cutoff, and capture a bargaining share of the jump. Under the recurring drop the stationary seller grid accumulates at M with positive mass at every point, so this deviation is available even if the jump is small. Thus continuity at M is an implication of equilibrium, not an assumption.

Continuity pins the candidate down to its slope, and the level conditions determine that slope. Define $Z(\mu) \equiv \alpha\beta(1 - c)(1 - \theta)(1 - H^*(L))$ and the marginal-value denominator

$$L(\mu) = \chi M \left[(1 - \beta) + \frac{\beta\mu}{1 + \mu} \right] + \frac{Z(\mu)[(\bar{m}_b - M) + \chi M]}{1 + \mu} - \frac{(1 - \beta + \mu)[\chi M + (1 - \chi)\bar{m}_s/(1 - c)]}{1 + \mu}. \quad (42)$$

Proposition 3 (Existence and uniqueness under the helicopter drop). *Assume $\mu > 0$ and $H^*(L) \in (0, 1)$, equivalently $\alpha\beta\theta(1 - c) > c(1 - \beta + \mu)$. If $L(\mu) > 0$, the class of two-branch linear pay-all equilibria under this policy, with value functions (35) and an arbitrary admissible jump Δ^v , contains exactly one equilibrium: it is continuous at the cutoff ($\Delta^v = 0$),*

with slopes

$$b^* = \frac{d}{L(\mu)}, \quad a^* = \chi \frac{d}{L(\mu)}, \quad (43)$$

anchor

$$(1 - \beta)v^*(0) = \frac{\beta a^* \mu M}{1 + \mu} + \frac{(1 - \beta + \mu)(b^* - a^*)}{(1 - c)(1 + \mu)} \bar{m}_s > 0, \quad (44)$$

and the stationary distribution of Lemma 3. The role choice is strict at every balance except $m = M$, where the agent is indifferent. If $L(\mu) \leq 0$, the class contains no equilibrium.

Proof. See Appendix B.2.3. □

The inequality $a^* < b^*$ is an *intertemporal wedge*: money is worth more when spent today (b^*) than when carried toward a future buyer state (a^*). Under the Friedman rule $\mu = \beta - 1$ the wedge would close ($a^* = b^*$). Under the drop it stays open, and because sellers now hold money, it enters the seller's continuation problem directly.

The anchor (44) is strictly positive, in contrast to the baseline normalization $v^*(0) = 0$. The first term is the transfer income of a fresh spender: a seller who has just spent everything receives $\mu M / (1 + \mu)$ next period, valued at the seller slope a^* . The second term is the rent on the balances sellers carry to the bargaining table, priced at the wedge $b^* - a^*$. The drop subsidizes the waiting state.

As $\mu \rightarrow 0$, $L(\mu) \rightarrow ZM[H^*(0)/(1 - H^*(0)) + \chi_0] > 0$, so small positive growth rates are admissible. In the limit the selected equilibrium converges to the baseline member with cutoff $\tilde{m} = M$ and no jump: b^* converges to the slope of Lemma 2 at that member, and $v^*(0) \rightarrow 0$.

5.3 Aggregate Equations and Welfare

Aggregate equations. Since welfare is the population average of values, the natural aggregate unknowns are the buyer-side and seller-side average values and the average joint surplus:

$$\bar{v}_b^* \equiv \frac{1}{1 - H^*(L)} \int_M^\infty v(m) dH^*(m), \quad \bar{v}_s^* \equiv \frac{1}{H^*(L)} \int_0^M v(m) dH^*(m),$$

$$\bar{J}^* \equiv \frac{1}{H^*(L)(1 - H^*(L))} \int_M^\infty \int_0^M J(m_b, m_s) dH^*(m_s) dH^*(m_b).$$

The all-agent transfer enters the aggregate system through three *inflation-drift terms*:

$$\Gamma_b \equiv \frac{\mu b^*}{1 + \mu} (\bar{m}_b - M), \quad \Gamma_s \equiv \frac{\mu a^*}{1 + \mu} (M - \bar{m}_s), \quad (45)$$

$$\Gamma_J \equiv \frac{(b^* - c a^*) \bar{m}_s}{1 + \mu}. \quad (46)$$

The term Γ_b is the inflation tax on the average buyer: buyers hold more than the fixed point M of the drift map g , so the transfer-cum-deflation erodes their real balances. The term Γ_s is the inflation subsidy to the average seller: sellers hold less than M , so the same operation lifts their real balances. The term Γ_J is the *reevaluation wedge*: seller balances are carried at the low marginal value $a^* < b^*$, but upon trade they exit bundled with the buyer's payment into the high-valuation buyer region, which raises the joint surplus.⁹ All three terms vanish at $\mu = 0$.

Proposition 4 (Aggregate equations under the helicopter drop). *Suppose the stationary equilibrium characterized in Proposition 3 exists. Write $K \equiv c + \theta - c\theta$ and*

$$\mathcal{D} \equiv (1 - \beta) - \alpha\beta(1 - c) \left[\frac{\theta H^*(L)}{c} - (1 - \theta)(1 - H^*(L)) \right]. \quad (47)$$

Given $H^(L)$ and the slope ratio χ from Lemma 4, the conditional mean balances (34), and the drift terms (45)–(46), the stationary aggregates satisfy*

$$(1 - \beta) \bar{v}_b^* = \alpha H^*(L) \tilde{\theta} \bar{J}^* - \beta \Gamma_b, \quad (48)$$

$$(1 - \beta) \bar{v}_s^* = -d + \alpha(1 - H^*(L))(1 - \tilde{\theta}) \bar{J}^* + \beta \Gamma_s, \quad (49)$$

$$\bar{J}^* = \frac{K}{c} \beta \left[(1 - c)(\bar{v}_b^* - \bar{v}_s^*) + \Gamma_J - (1 - c)(\Gamma_b + \Gamma_s) \right]. \quad (50)$$

If $\mathcal{D} \neq 0$, the system has the triangular representation

$$\bar{J}^* = \frac{K\beta}{c\mathcal{D}} \left[(1 - c)d + (1 - \beta)\Gamma_J - (1 - c)(\Gamma_b + \Gamma_s) \right]. \quad (51)$$

⁹The wedge Γ_J is not simply the average of the per-match wedge in (37). It bundles the revaluation gain $(b^* - a^*)\bar{m}_s/(1 + \mu)$ with the terms that the seller's carried balance contributes to the bargaining threat points, $(1 - c)a^*\bar{m}_s - (1 - c)\mu a^*\bar{m}_s/(1 + \mu)$; summing the three terms gives $(b^* - c a^*)\bar{m}_s/(1 + \mu)$. Defining Γ_J this way removes the zero-money drift $\mu a^*M/(1 + \mu)$ from the system and makes (50) symmetric in Γ_b and Γ_s ; see the proof of Proposition 4.

Given $\bar{J}$, the buyer and seller average values follow directly from (48) and (49).

Proof. See Appendix B.3.1. □

The three equations have a transparent reading. Equation (48) says that buyers earn their expected share of the joint surplus and pay the inflation tax Γ_b . Equation (49) says that sellers pay the sunk cost, earn their expected share, and receive the drift subsidy Γ_s . Equation (50) says that the average surplus is pinned down by the buyer–seller value gap together with the redistribution terms. Under the all-agent drop, all three drift terms are active, and they are the source of the non-neutrality of a uniform helicopter drop. Setting $\Gamma_b = \Gamma_s = \Gamma_J = 0$ in (48)–(50) yields a drift-free benchmark in which the injection has no direct effect and μ operates only through the seller mass. Substituting $H^*(L)$ from Lemma 4 into (47) gives $\mathcal{D} = Z(\mu) - \mu$.

Welfare. Write $H \equiv H^*(L)$, making the dependence on μ explicit as $H(\mu)$ where it matters, and let $W(\mu)$ denote stationary welfare under money growth μ . Welfare is the population average of values and, equivalently, the discounted production surplus net of the sunk cost:

$$W(\mu) \equiv H \bar{v}_s^* + (1 - H) \bar{v}_b^* = \frac{\alpha H(1 - H)(1 - c)\bar{x}^* - H d}{1 - \beta}. \quad (52)$$

Substituting the aggregate equations (48)–(49) into the population average gives the equivalent welfare identity

$$W(\mu) = \frac{1}{1 - \beta} \left[\alpha H(1 - H) \bar{J}^* - H d + \beta (H \Gamma_s - (1 - H) \Gamma_b) \right]. \quad (53)$$

The two expressions agree because averaging (37) gives $(1 - c)\bar{x}^* = \bar{J}^* - \beta(b^* - a^*)\bar{m}_s/(1 + \mu)$, while the stationary distribution satisfies

$$(1 - H) \Gamma_b - H \Gamma_s = \alpha H(1 - H) \frac{(b^* - a^*)\bar{m}_s}{1 + \mu}. \quad (54)$$

Thus the drift terms exactly absorb the per-match revaluation wedge. At $\mu = 0$ they vanish and (53) reduces to the baseline welfare of Section 3.

Equation (53) separates four routes through which a marginal recurring drop affects

welfare. To distinguish zero-growth evaluations from time indices, we write levels as $X(0)$ and right derivatives as $X'(0^+)$. In particular, $H'(0^+) = H^*(0)/(1 - \beta)$.

Proposition 5 (Marginal welfare decomposition). *Suppose (30) holds. Along the recurring-drop equilibrium branch, $\Gamma_b(0) = \Gamma_s(0) = 0$ and*

$$\begin{aligned} (1 - \beta)W'(0^+) = & \underbrace{\alpha(1 - 2H^*(0))H'(0^+)\bar{J}^*(0)}_{\text{match-count effect}} - \underbrace{H'(0^+)d}_{\text{sunk-cost effect}} \\ & + \underbrace{\alpha H^*(0)(1 - H^*(0))(\bar{J}^*)'(0^+)}_{\text{surplus-per-match effect}} + \underbrace{\beta[H^*(0)\Gamma'_s(0^+) - (1 - H^*(0))\Gamma'_b(0^+)]}_{\text{direct drift effect}}. \end{aligned} \quad (55)$$

Moreover, these four effects sum to a strictly positive response:

$$W'(0^+) > 0. \quad (56)$$

Proof. Along the equilibrium branch in the statement, the hypotheses of Proposition 3 hold for all sufficiently small $\mu > 0$. Hence the aggregate identities of Proposition 4, including (53), apply when the right derivative at zero is taken.

Along the recurring-drop branch, differentiate (53) from the right at $\mu = 0$. Because $\Gamma_b(0) = \Gamma_s(0) = 0$, the product-rule terms involving $H'(0^+)$ and the levels of the drift terms vanish, and the remaining terms are exactly the four effects in (55). It remains to evaluate them and to sign their sum.

The seller-mass formula (40) and the conditional means (34) give the zero-growth responses

$$\begin{aligned} H'(0^+) &= \frac{H^*(0)}{1 - \beta}, & \bar{m}_s(0) &= 0, & \bar{m}'_s(0^+) &= \frac{M}{\alpha(1 - H^*(0))}, & \bar{m}_b(0) &= \frac{M}{1 - H^*(0)}, \\ \Gamma'_b(0^+) &= \frac{b^*(0)MH^*(0)}{1 - H^*(0)}, & \Gamma'_s(0^+) &= a^*(0)M, & \Gamma'_J(0^+) &= [b^*(0) - ca^*(0)]\bar{m}'_s(0^+), \end{aligned} \quad (57)$$

so that the direct drift effect is

$$\beta[H^*(0)\Gamma'_s(0^+) - (1 - H^*(0))\Gamma'_b(0^+)] = -\beta H^*(0)[b^*(0) - a^*(0)]M < 0, \quad (58)$$

the inequality because $b^*(0) > 0$ and $a^*(0) = \chi(0)b^*(0)$ with $0 < \chi(0) < 1$. Differentiating the surplus solution (51) gives the response bundled in the third effect,

$$\begin{aligned} (\bar{J}^*)'(0^+) &= -\frac{\mathcal{D}'(0^+)}{\mathcal{D}(0)}\bar{J}^*(0) + \frac{K\beta}{c\mathcal{D}(0)}\left[(1-\beta)\Gamma'_J(0^+) - (1-c)(\Gamma'_b(0^+) + \Gamma'_s(0^+))\right], \\ \mathcal{D}'(0^+) &= -\alpha\beta(1-c)\frac{K}{c}H'(0^+), \quad \mathcal{D}(0) = Z > 0, \end{aligned} \quad (59)$$

which combines the seller-mass, revaluation, and drift responses and completes the evaluation of (55).

For the sign of the total, note that (51) at zero gives $\bar{J}^*(0) = K\beta(1-c)d/(cZ)$, while (57) in (42) gives

$$L(0) = ZM \left[\frac{H^*(0)}{1-H^*(0)} + \chi(0) \right] > 0. \quad (60)$$

Write $\eta_s \equiv c(1-\theta)(1-H^*(0))$ and $\eta_b \equiv \theta(1-c)H^*(0)$, two abbreviations local to this proof, so that (40) and (41) read $\alpha = c(1-\beta)/[\beta\theta(1-c)H^*(0)]$ and $\chi(0) = \eta_s/(\eta_s + \eta_b)$. Substituting (57) and (59) into (55), and using $b^*(0) = d/L(0)$ with (60), reduces the sum of the four effects, by a routine rational-function computation that we verified symbolically, to

$$W'(0^+) = \frac{dH^*(0)\theta}{c(1-\theta)(1-\beta)^2} \left[1 + \frac{\beta\eta_b(\eta_s\eta_b + KH^*(0)^2\theta)}{c\eta_s(\eta_s + H^*(0)\eta_b)} \right] > 0, \quad (61)$$

every factor being strictly positive. This proves (56). $\square$

The match-count effect has the sign of $1 - 2H^*(0)$, whereas the sunk-cost effect and the direct drift effect (58) are negative: the buyers' inflation tax exceeds the sellers' drift subsidy by the wedge, times the money stock, times the seller mass. The surplus-per-match effect bundles the seller-mass response with the revaluation and drift responses through (59); it has no independently obvious sign, and the accounting decomposition alone does not determine the sign of the total; the closed form (61) does.

As $\mu \downarrow 0$, the recurring-drop branch selects the member of the baseline family with cutoff M and no jump. Because $W(0)$ is common to the whole baseline family, both the benchmark and the right derivative along the policy branch are well defined.

Proposition 5 is the local policy result: the positive total response does not require each matching, sunk-cost, and redistribution channel to have the same sign. The next proposition

extends the derivative claim itself to every admissible growth rate. Let

$$\mu_H \equiv \frac{\alpha\beta\theta(1-c)}{c} - (1-\beta) > 0, \quad (62)$$

so that $H(\mu_H) = 1$, and define

$$\mathcal{M} \equiv \{\mu > 0 : H(\mu) < 1 \text{ and } L(\mu) > 0\}, \quad \bar{\mu} \equiv \sup\{m > 0 : (0, m) \subseteq \mathcal{M}\}. \quad (63)$$

Proposition 6 (Welfare under a constant helicopter drop). *Suppose (30) holds. Then:*

(i) $\bar{\mu} > 0$ and

$$\frac{dW(\mu)}{d\mu} > 0 \quad \text{for every } \mu \in (0, \bar{\mu}). \quad (64)$$

(ii) *The admissible component ends at the first zero of L , strictly before $H = 1$:*

$$0 < \bar{\mu} < \mu_H, \quad L(\bar{\mu}) = 0, \quad H(\bar{\mu}) < 1. \quad (65)$$

Moreover,

$$b^* \longrightarrow +\infty, \quad W(\mu) \longrightarrow +\infty \quad \text{as } \mu \uparrow \bar{\mu}. \quad (66)$$

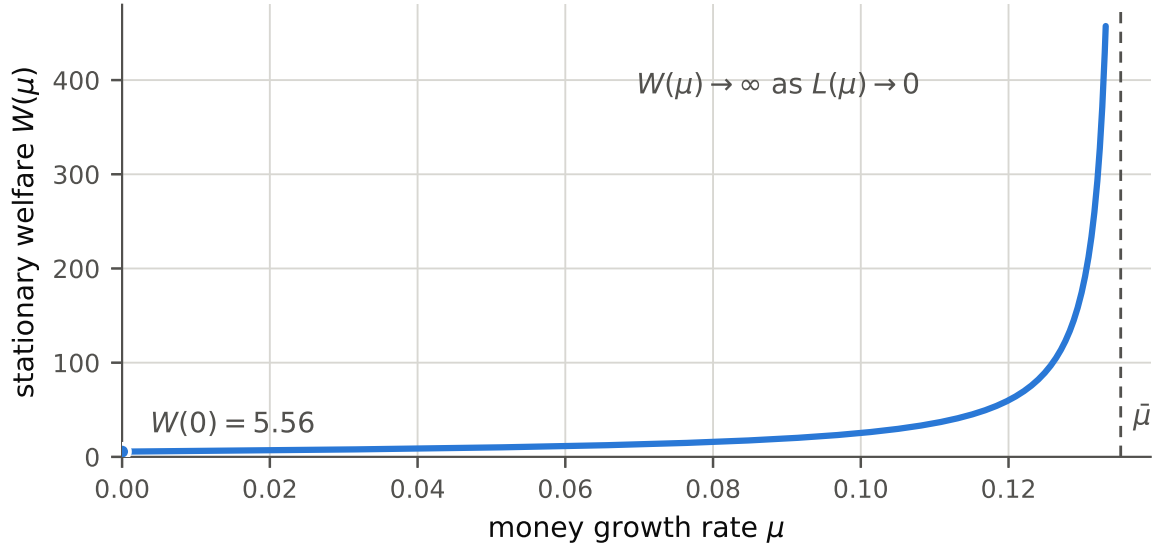
Proof. See Appendix C.1. □

The proof is computer-assisted. After a change of variables, the sign of $dW(\mu)/d\mu$ reduces to the positivity of a polynomial on the unit cube. Appendix C.1 verifies this positivity using an exact Bernstein-basis certificate and provides the program.

The restriction to positive growth comes from the uniform incidence of the drop. Appendix A studies instead a transfer targeted to buyers. Within the construction developed there, this policy can implement deflation. Under this drift-free benchmark, μ affects welfare only through the seller mass; welfare remains increasing in μ , and the Friedman rule is reached only as a degenerate limit with no sellers and no trade.

Figure 3 illustrates Proposition 6 for one calibration over the whole admissible range.

Figure 3: Stationary welfare under constant money growth.



Note: The parameters are $(\alpha, \beta, \theta, c) = (0.8, 0.9, 0.5, 0.5)$ and $M = d = 1$. The admissible range ends at $\bar{\mu} = 0.1352$.

6 Monetary Policies with One-Time Injection

We now study a one-time helicopter drop. Through period 0 the economy is in an admissible stationary pay-all equilibrium of Section 4, with money stock M , cutoff $\tilde{m}$, and jump Δ_0^v . At the start of period 1, every agent unexpectedly receives

$$\tau = \mu M > 0. \tag{67}$$

The money stock is $(1 + \mu)M$ thereafter and there are no further transfers. We first construct the exact on-path transition and then state the conditions under which bargaining, role choice, and participation are valid on path. A narrow off-path limitation remains and is stated explicitly below.

6.1 The candidate and the injection pool

Write $H_0 \equiv H^*(0)$ for the stationary seller mass at zero growth and continue to use $K = c + \theta - c\theta$. The identities

$$\frac{\alpha H_0 \theta \beta (1 - c)}{c} = 1 - \beta, \quad Z \equiv \alpha \beta (1 - c)(1 - \theta)(1 - H_0), \quad (68)$$

$$\chi \equiv \frac{\alpha \beta (1 - H_0)(1 - \theta)}{1 - \beta + \alpha \beta (1 - H_0)(1 - \theta)}, \quad \gamma_3 \equiv 1 - \alpha(1 - H_0) \in (0, 1) \quad (69)$$

follow from the baseline equilibrium. The initial member has slopes $a^* = \chi b^* < b^*$ and satisfies

$$Z \left[a^* \tilde{m} + \Delta_0^v + b^* \left(\frac{M}{1 - H_0} - \tilde{m} \right) \right] = d, \quad (70)$$

or equivalently

$$b^* = \frac{d/Z - \Delta_0^v}{M/(1 - H_0) - (1 - \chi)\tilde{m}} > 0. \quad (71)$$

The initial member also satisfies its own pay-all bound $c\Delta_0^v \leq (b^* - ca^*)\tilde{m}$.

Throughout this section, we maintain this admissible pre-injection equilibrium, including (30), its pay-all bound, and the identities (68)–(71). Additional restrictions are stated proposition by proposition.

The injection makes the economy nonstationary, so Definition 3 does not apply directly; we use the same two-branch form at each date. No transfer follows the injection, so balances again do not drift between trades, and holding the cutoff at its pre-injection value $\tilde{m}$ is part of the construction. We look for a transition with the literal cutoff jump as a state variable: for $t \geq 1$,

$$v_t(m) = \begin{cases} v_{0,t} + a_t m, & m < \tilde{m}, \\ v_{0,t} + a_t \tilde{m} + \Delta_t^v + b_t(m - \tilde{m}), & m \geq \tilde{m}. \end{cases} \quad (72)$$

The first premise keeps the transferred balances, including those used in the deviation checks, strictly below the fixed cutoff:

$$(R') \quad 2\tau < \tilde{m}. \quad (73)$$

The buyer-slope recursion makes b_t constant but does not determine its level. We select this constant at its pre-announcement value b^* . The unique bounded seller-slope sequence is then $a_t = a^* = \chi b^*$ for every $t \geq 1$. Under the candidate pay-all path and (R'), sellers hold either 0 or τ , buyers hold at least $\tilde{m} + \tau$, and the seller mass remains H_0 . If $H_t(\tau)$ is the mass of sellers still holding the injection, then

$$H_{t+1}(\tau) = \gamma_3 H_t(\tau), \quad H_1(\tau) = H_0. \quad (74)$$

Thus, with $h_t \equiv H_t(\tau)/H_0 = \gamma_3^{t-1}$,

$$\overline{m}_{s,t} = h_t \tau, \quad \overline{m}_{b,t} = \frac{(1 + \mu)M - H_0 h_t \tau}{1 - H_0}. \quad (75)$$

The injection creates a temporary mass of τ -sellers. A seller leaves that pool when it trades and is replaced by a zero-balance seller, so the pool decays geometrically even though the total seller mass stays fixed. Figure 4 illustrates this redistribution.

6.2 The three-equation transition system

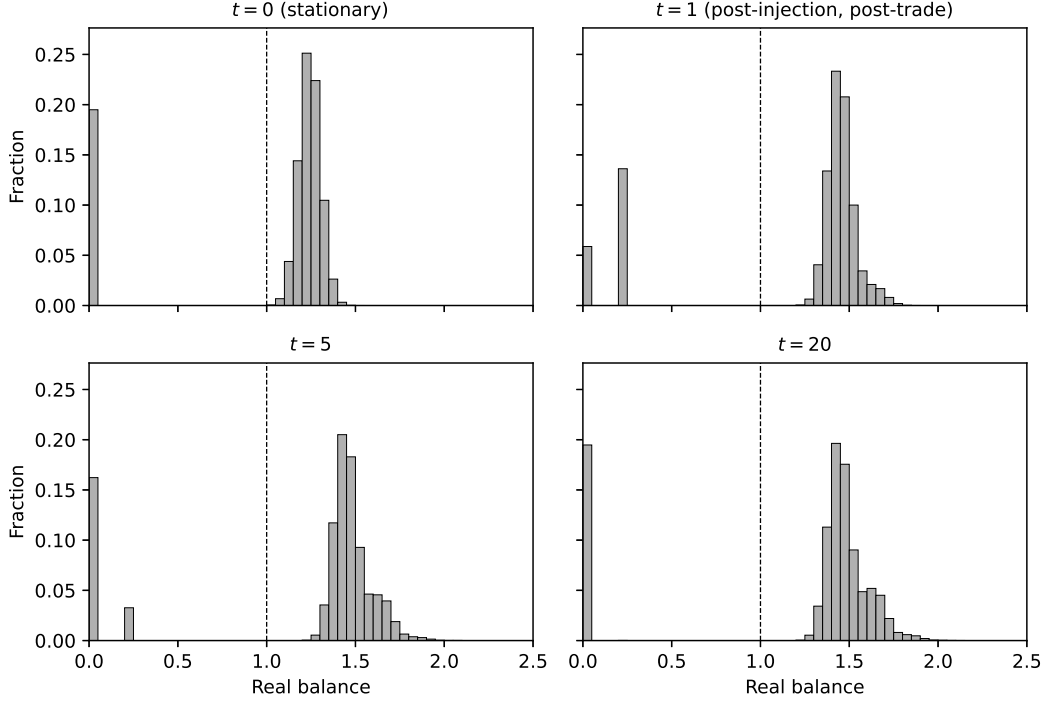
In an on-path meeting, a buyer holds $m_b \geq \tilde{m} + \tau$ and a seller holds $m_s \in \{0, \tau\}$. Both post-trade balances lie on their intended branches. The continuation increments and bargained quantity are

$$\begin{aligned} \Delta_{b,t} &\equiv a^* \tilde{m} + \Delta_{t+1}^v + b^*(m_b - \tilde{m}), \\ \Delta_{s,t} &= \Delta_{b,t} + (b^* - a^*)m_s, \end{aligned} \quad (76)$$

$$x_t(m_b, m_s) = \frac{K\beta}{c} \Delta_{b,t} + \frac{\theta\beta}{c} (b^* - a^*)m_s. \quad (77)$$

Substitution into the buyer and seller Bellman equations eliminates the coefficients of individual balances. Together with the pool law, the remaining constants form the three

Figure 4: Transition of the money-holdings distribution after a one-time helicopter drop.



Note: Simulated with $\alpha = 0.3$, $\beta = 0.95$, $c = 0.5$, $\theta = 0.9$, $\mu = 0.2$, and $M = 1$. The redistribution is payoff-independent and holds for any admissible initial member, so condition (I) below is not needed for this figure. The dashed line marks the cutoff. At $t = 1$ all sellers hold $\tau = 0.2$. By $t = 5$ the mass at τ has fallen to $\gamma_3^4 H_0 \approx 0.065$, and by $t = 20$ the distribution is close to its stationary form, with the buyer distribution shifted right to reflect the larger money stock.

equations

$$\Delta_t^v + v_{0,t} = \Delta_{t+1}^v + \beta v_{0,t+1} + K_0 \tau h_t, \quad (78)$$

$$v_{0,t} = Z \left[a^* \tilde{m} + \Delta_{t+1}^v + b^* \left(\frac{(1 + \mu)M - H_0 h_t \tau}{1 - H_0} - \tilde{m} \right) \right] + \beta v_{0,t+1} - d, \quad (79)$$

$$h_{t+1} = \gamma_3 h_t, \quad h_1 = 1, \quad (80)$$

where

$$K_0 \equiv \frac{\alpha H_0 \theta \beta (b^* - a^*)}{c} = \frac{(1 - \beta)(b^* - a^*)}{1 - c}. \quad (81)$$

These equations impose the prescribed buyer and seller branches but do not yet establish that agents prefer those roles.

In matrix form, the system is

$$\begin{aligned}
& \begin{bmatrix} 1 & \beta & 0 \\ Z & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta_{t+1}^v \\ v_{0,t+1} \\ h_{t+1} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & -K_0\tau \\ 0 & 1 & Zb^*H_0\tau/(1-H_0) \\ 0 & 0 & \gamma_3 \end{bmatrix} \begin{bmatrix} \Delta_t^v \\ v_{0,t} \\ h_t \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ d - Z[b^*(1+\mu)M/(1-H_0) - (b^* - a^*)\tilde{m}] \\ 0 \end{bmatrix}.
\end{aligned} \tag{82}$$

At $h = 0$, the steady-state equations give

$$v_{0,\infty} = 0, \quad \Delta_\infty^v = \frac{d}{Z} - \frac{b^*(1+\mu)M}{1-H_0} + (b^* - a^*)\tilde{m}. \tag{83}$$

Subtracting the initial family condition (70) gives the exact drain in the jump,

$$\Delta_\infty^v = \Delta_0^v - \frac{b^*\tau}{1-H_0}. \tag{84}$$

The terminal member therefore belongs to the stationary family only under the second premise

$$(T) \quad \Delta_\infty^v \geq 0, \quad \text{equivalently} \quad \Delta_0^v \geq \frac{b^*\tau}{1-H_0}. \tag{85}$$

In particular, the continuous initial member does not admit a positive fixed-cutoff transition of this form.

Centering the two forward variables at their terminal values yields

$$\begin{aligned}
& \begin{bmatrix} 1 & \beta & 0 \\ Z & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta_{t+1}^v - \Delta_\infty^v \\ v_{0,t+1} \\ h_{t+1} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & -K_0\tau \\ 0 & 1 & Zb^*H_0\tau/(1-H_0) \\ 0 & 0 & \gamma_3 \end{bmatrix} \begin{bmatrix} \Delta_t^v - \Delta_\infty^v \\ v_{0,t} \\ h_t \end{bmatrix}.
\end{aligned} \tag{86}$$

Proposition 7 (The one-time transition and its closed form). *Assume (R') and (T). The characteristic roots of (86) are*

$$\frac{1}{1-Z} > 1, \quad \frac{1}{\beta} > 1, \quad \gamma_3 \in (0, 1). \tag{87}$$

There is one predetermined variable, $h_1 = 1$, and one stable root, so the convergent on-path solution is unique. It is

$$\begin{bmatrix} \Delta_t^v \\ v_{0,t} \\ h_t \end{bmatrix} = \begin{bmatrix} \Delta_\infty^v \\ 0 \\ 0 \end{bmatrix} + \gamma_3^{t-1} \begin{bmatrix} \varphi_\Delta \\ \varphi_v \\ 1 \end{bmatrix}, \tag{88}$$

where

$$\varphi_\Delta = \frac{K_0\tau + Zb^*H_0\tau/(1-H_0)}{1 - \gamma_3 + Z\gamma_3} > 0, \tag{89}$$

$$\varphi_v = \frac{Z[\gamma_3\varphi_\Delta - b^*H_0\tau/(1-H_0)]}{1 - \beta\gamma_3}. \tag{90}$$

Hence Δ_t^v falls geometrically to $\Delta_\infty^v \geq 0$. The system proves convergence of the candidate path; Proposition 9 below supplies its incentive and scope statements.

Proof. See Appendix B.4. □

6.3 Output, prices, and welfare

Proposition 8 (Output, prices, and welfare along the transition). *Assume (R') and (T).*

Let $\bar{x}_t$ be average production per trade and $\hat{x}_1 \equiv \bar{x}_1 - \bar{x}_\infty$. Then

$$\bar{x}_t = \bar{x}_\infty + \hat{x}_1 \gamma_3^{t-1}, \quad \bar{x}_\infty = \frac{K\beta}{c} \frac{d}{Z}, \quad (91)$$

$$\hat{x}_1 = \frac{\beta}{c} \left[K \frac{(1 - \beta\gamma_3)\varphi_v}{Z} + \theta(b^* - a^*)\tau \right], \quad (92)$$

$$Y_t = \alpha H_0 (1 - H_0) \bar{x}_t. \quad (93)$$

The long-run real allocation equals the pre-drop allocation. With buyers paying all, the price index is

$$p_t = \frac{[(1 + \mu)M - H_0 \tau \gamma_3^{t-1}]/(1 - H_0)}{\bar{x}_t}, \quad p_{\text{pre}} = \frac{M/(1 - H_0)}{\bar{x}_\infty}, \quad p_\infty = (1 + \mu)p_{\text{pre}}. \quad (94)$$

Define the selection-free bounds

$$\hat{x}_U \equiv \mu(1 - H_0)\bar{x}_\infty > 0, \quad \hat{x}_L \equiv -\frac{\mu H_0 \bar{x}_\infty}{1 + \mu} < 0. \quad (95)$$

The price path has three cases:

- (a) *If $\hat{x}_L < \hat{x}_1 < \hat{x}_U$, p_1 jumps above p_{pre} and then rises to p_∞ .*
- (b) *If $\hat{x}_1 > \hat{x}_U$, the economy experiences impact deflation, followed by a monotone rise to p_∞ .*
- (c) *If $\hat{x}_1 < \hat{x}_L$, p_1 overshoots p_∞ and then falls monotonically.*

Finally, discounted welfare relative to the terminal allocation is

$$\Delta W = \frac{(1 - c)\alpha H_0 (1 - H_0)}{1 - \beta\gamma_3} \hat{x}_1, \quad \text{sign}(\Delta W) = \text{sign}(\hat{x}_1). \quad (96)$$

Proof. See Appendix B.4. □

The impact response reflects the bargaining wedge. The buyer's balance rises by τ , valued at b^* , while the seller's threat point rises by the same amount, valued at a^* . Both components of $\hat{x}_1$ are proportional to $b^*\tau$, so the selection scales out of the sign comparisons. The on-path dynamics alone leave the sign open; the incentive condition below closes it.

6.4 On-path incentives and scope

The bargaining problem at date t uses the continuation jump Δ_{t+1}^v . Because the jump falls over time, the largest such jump is Δ_2^v . The pay-all premise is

$$(C) \quad c\Delta_2^v \leq (b^* - ca^*)\tilde{m}, \quad \Delta_2^v = \Delta_\infty^v + \gamma_3\varphi_\Delta. \quad (97)$$

For an on-path seller, waiting creates two opposing effects. A trade can push an injection holder into the buyer region, allowing the seller to capture the revaluation wedge, but waiting also exposes the agent to the erosion of the buyer-position premium. Their difference is

$$v_{S,t}(m) - v_{B,t}(m) = \underbrace{K_0 h_t \tau}_{\text{capture}} + \underbrace{(\Delta_{t+1}^v - \Delta_t^v)}_{\text{erosion} < 0} = h_t [K_0 \tau - (1 - \gamma_3)\varphi_\Delta], \quad (98)$$

for $m \in \{0, \tau\}$.

Proposition 9 (On-path validity and scope of the one-time transition). *Assume (R') , (T) , and (C) . Then:*

- (i) *Pay-all is optimal and generates strictly positive surplus in every on-path meeting.*
- (ii) *For every finite date,*

$$\text{sign}(v_{S,t}(m) - v_{B,t}(m)) = \text{sign}(v_{0,t}) = \text{sign}(\beta\theta\gamma_3(1 - \chi) - c), \quad m \in \{0, \tau\}. \quad (99)$$

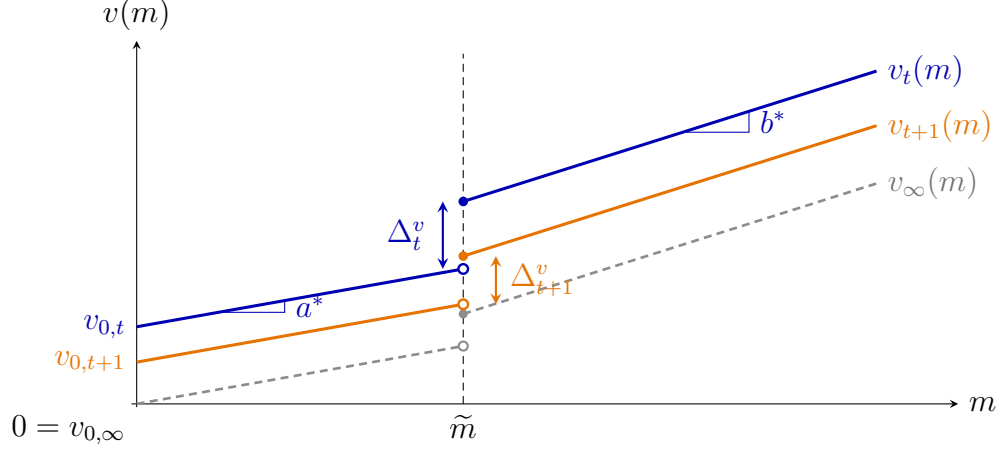
$$(I) \quad \beta\theta\gamma_3(1 - \chi) > c, \quad (100)$$

Under (I), all on-path sellers strictly prefer S, all buyers strictly prefer B, and every on-path value is positive. Under the reverse strict inequality, $v_t(0) < 0$, so permanent idling yields a profitable deviation and rules out a transition of this form.

- (iii) *Under (I), $\varphi_v > 0$ and $\hat{x}_1 > 0$. Output and welfare rise on impact and decay geometrically to their stationary levels; case (c) of Proposition 8 is impossible.*
- (iv) *Even under (I), at every finite date the prescribed seller assignment fails on a nonempty subinterval of $[\tilde{m} - \tau, \tilde{m})$. Hence the construction is an exact on-path transition, not a fully verified pointwise equilibrium.*

Proof. See Appendix B.4. □

Figure 5: Candidate value profiles during the one-time transition.



Note: Under condition (I), $v_{0,t+1} < v_{0,t}$ and $\Delta_{t+1}^v < \Delta_t^v$. The cutoff $\tilde{m}$ and the slopes a^* and b^* are time invariant. Repeating the movement shown in the figure gives the terminal profile with $v_{0,\infty} = 0$ and $\Delta_\infty^v \geq 0$. These are the candidate on-path profiles; the off-path limitation in Proposition 9(iv) remains.

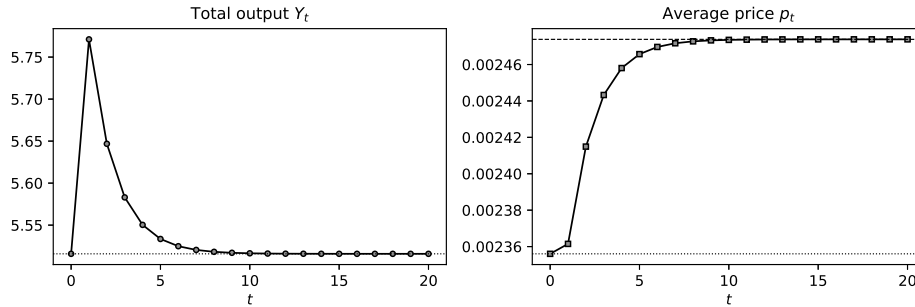
Condition (I) is a substantive restriction, most naturally satisfied when c is low and θ is high. Reducing τ does not change its sign because capture and erosion in (98) scale together. The equality case is left aside.

The limitation in part (iv) is entirely off path. No on-path balance lies in the problematic window, and all role conditions outside it are verified. The window has width at most τ . Although it remains nonempty at every finite date, the role-gap violation carries the common factor $h_t = \gamma_3^{t-1}$, so its magnitude vanishes geometrically as the injection is absorbed.

Under (I), the candidate value profile also converges geometrically. The cutoff and the two slopes remain fixed, while the anchor $v_{0,t}$ falls to zero and the jump Δ_t^v narrows to Δ_∞^v . Figure 5 illustrates this movement.

Gradual adjustment is possible because the terminal zero-growth equilibrium belongs to the two-parameter family of Section 4. The jump supplies one level variable per period, and the path moves from $(\tilde{m}, \Delta_0^v)$ to $(\tilde{m}, \Delta_\infty^v)$ while the cutoff remains fixed. Under recurring money growth, the endpoint is unique and this freedom disappears. Keeping b^* at its pre-announcement level is an equilibrium selection. If instead the value of money falls at the announcement and the economy converges to $b' = b^*/(1 + \mu)$, every real object is rescaled by $1/(1 + \mu)$ and all signs, including (99), are unchanged. The terminal jump becomes $\Delta_\infty^v = \Delta_0^v - (b^* - a^*)\tilde{m}\mu/(1 + \mu)$, so the positive initial jump required by (T) is smaller but

Figure 6: Time paths of total output and the average price after a one-time helicopter drop.



Note: Computed from (91) and (94) at $(\alpha, \beta, \theta, c) = (0.5, 0.9, 0.95, 0.10)$, with $\mu = 0.05$, $M = d = 1$, $\tilde{m} = 0.15$, and Δ_0^v at the bound in (T). Conditions (R'), (T), (C), and (I) hold. Output jumps above its stationary level and then decays at rate γ_3^{t-1} as the injection moves from sellers to buyers. The price rises from p_{pre} toward $p_{\infty} = (1 + \mu)p_{\text{pre}}$, illustrating case (a) of Proposition 8.

does not disappear.

7 Conclusion

This paper has developed a divisible-money search-and-bargaining model without a centralized market to analyze welfare effects of monetary policies. By introducing the pay-all equilibrium and exploiting linearity in preferences and technology, we solve the stationary policy problem and reduce the one-time injection to an exact closed-form on-path transition.

The equilibrium existence condition was shown to be strikingly simple: the presence of a positive measure of buyers is both necessary and sufficient. Our analysis identifies two central mechanisms through which monetary policy affects welfare. The distribution of money holdings changes the seller mass, and hence matching and sunk costs, while redistribution between buyers and sellers changes bargaining surplus because their marginal values of money differ. Stationary welfare is strictly increasing in the money-growth rate at every admissible rate. For a one-time injection, output and welfare rise on impact under an explicit condition on primitives. The resulting closed form is exact on path, but the prescribed role assignment remains incomplete on a narrow, never-visited off-path window below the cutoff.

In relation to the literature, our model can be viewed as a divisible-money counterpart of Trejos and Wright (1995), complementing prior numerical studies (Molico, 2006; Chiu and Molico, 2021) and Lagos-Wright-type analyses (Chiu and Molico, 2010, 2011). Unlike

these contributions, our framework provides a closed-form stationary welfare comparison and closed-form on-path dynamics, while clarifying how transfer incidence and money-holding redistribution shape policy effects. More broadly, our results are closely related to the redistribution channel emphasized in recent HANK models.

Future research may enrich the framework by incorporating capital accumulation or non-linearity in preferences and technologies. Nonetheless, the central message remains: when exchange is decentralized and money holdings are heterogeneous, the welfare effects of monetary policy depend on both the incidence of transfers and the endogenous distribution of balances.

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Appendix

A The Targeted Transfer

The helicopter drop of Section 5 reaches every agent, so negative growth is infeasible: an agent holding no money cannot pay a lump-sum tax. This appendix studies the natural alternative, a transfer targeted to buyers. At the start of each period the government pays the transfer only to agents whose balance is at least the role cutoff; in equilibrium these are exactly the buyers, and each receives an equal share of the aggregate injection μM_{t-1}^n . Sellers are never taxed, so any growth rate $\mu \in (\beta - 1, \bar{\mu}^B)$ is feasible, where $\bar{\mu}^B > 0$ is the unique solution of $Z(\mu) = \mu$.

The targeted transfer is the drift-free benchmark of Section 5.3: the policy under which only the seller mass changes with μ . Sellers receive nothing and hold no balances, so $\bar{m}_s = 0$ and the drift terms Γ_s and Γ_J of (45)–(46) are zero. On the buyer side, the no-trade map is now $m \mapsto (m + \mu \bar{m}_b)/(1 + \mu)$, whose fixed point is the buyers' own average balance $\bar{m}_b = M/(1 - H^{B*})$ rather than M : the transfer returns to the average buyer exactly what inflation takes, so the buyer drift term is zero as well. Redistribution occurs only within the buyer group, and since the joint surplus is affine in the buyer's balance, it nets out of every average. The injection therefore has no direct effect at any μ , and μ operates only through the seller mass.

The equilibrium objects follow from the equations of Section 5. The seller mass and the slope ratio take the same closed forms as under the drop:

$$H^{B*} = \frac{c(1 - \beta + \mu)}{\alpha\beta\theta(1 - c)}, \quad a^* = \chi b^*, \quad (\text{A.1})$$

with χ and $Z(\mu)$ evaluated at this common seller mass. With all drift terms zero, the aggregate system (48)–(50) of Proposition 4 applies verbatim, $\mathcal{D} = Z(\mu) - \mu$, and (51) reduces to

$$\bar{J}^{B*} = \frac{K\beta(1 - c)d}{c(Z(\mu) - \mu)}. \quad (\text{A.2})$$

Since $\bar{m}_s = 0$, the per-match identity (37) gives $(1 - c)\bar{x}^{B*} = \bar{J}^{B*}$, and the welfare formula

(52) becomes

$$W^{B*} = \frac{\alpha H^{B*}(1 - H^{B*})\bar{J}^{B*} - H^{B*}d}{1 - \beta}. \quad (\text{A.3})$$

Welfare is the same for every member of the family, and only H^{B*} and the free-entry factor $Z(\mu) - \mu$ move with μ .

Proposition A.1 (The targeted transfer and the Friedman rule). *Suppose (30) holds. Then*

$$\frac{dW^{B*}}{d\mu} = \frac{d(1 - H^*(0))[\theta H^*(0) + c(1 - \theta)]}{\theta H^*(0)(Z(\mu) - \mu)^2} > 0 \quad \text{for every } \mu \in (\beta - 1, \bar{\mu}^B). \quad (\text{A.4})$$

As $\mu \downarrow \beta - 1$, $\chi \rightarrow 1$, so $a^* = b^*$, while $H^{B*} \rightarrow 0$.

Proof. Step 1: The closed forms (A.1). On path a seller holds 0: a buyer who pays all falls below the threshold and receives no transfer until it next trades. For a buyer with $m \geq \tilde{m}$, the no-trade balance is $g^B(m) = (m + \mu\bar{m}_b)/(1 + \mu)$, so the Nash condition and the buyer Bellman equation give the two expressions

$$J(m) = \frac{K\beta(1 - c)}{c} [v(g^B(m)) - v(0)], \quad J(m) = \frac{1}{\alpha\tilde{\theta}H^{B*}} [v(m) - \beta v(g^B(m))],$$

both affine in m . Matching the coefficients on m gives the identity (B.28), hence H^{B*} . The no-trade maps of both policies have slope $1/(1 + \mu)$ and differ only in their intercepts, so the seller-side matching of Lemma 4 is unchanged and gives $a^* = \chi b^*$.

Step 2: The drift-free system. Since $\bar{m}_s = 0$, the terms Γ_s and Γ_J vanish. Averaging the buyer Bellman equation over the buyer distribution, the depreciation and the injection cancel exactly at the mean,

$$\int_{\tilde{m}}^{\infty} \frac{m + \mu\bar{m}_b}{1 + \mu} \frac{dH(m)}{1 - H^{B*}} = \frac{\bar{m}_b + \mu\bar{m}_b}{1 + \mu} = \bar{m}_b,$$

so no buyer drift term appears. The aggregates therefore satisfy (48)–(50) with $\Gamma_b = \Gamma_s = \Gamma_J = 0$; with $\mathcal{D} = Z(\mu) - \mu$, (51) gives (A.2), and (52) gives (A.3).

Step 3: Monotonicity. Write $u \equiv 1 - \beta + \mu$. Substituting (A.1) into (A.2) and (A.3),

$$Z(\mu) - \mu = A - \frac{K}{\theta}u, \quad W^{B*} = \frac{d}{1 - \beta} \frac{C}{\alpha\beta\theta(1 - c)} \frac{u}{A - (K/\theta)u},$$

where $A \equiv \alpha\beta(1-c)(1-\theta) + (1-\beta)$ and $C \equiv \alpha\beta\theta(1-c) - c(1-\beta) = \alpha\beta\theta(1-c)(1-H^*(0))$ are free of μ . Under (30), $C > 0$, and on $(\beta-1, \bar{\mu}^B)$ both u and $(A - (K/\theta)u)^{-1}$ are positive and strictly increasing in μ , so W^{B*} is strictly increasing. Differentiating and eliminating α through $\alpha\beta\theta(1-c) = c(1-\beta)/H^*(0)$ gives (A.4), a routine rational-function computation that we verified symbolically. Finally, as $\mu \downarrow \beta-1$, $u \rightarrow 0$, so $H^{B*} \rightarrow 0$ and the denominator of χ converges to its numerator, giving $\chi \rightarrow 1$. $\square$

The limit in the proposition is the Friedman rule. Deflation at the rate of time preference makes carrying money across periods costless, and the intertemporal wedge closes: the marginal value of money is equalized across the seller and buyer states, the counterpart of a zero nominal interest rate in Lagos and Wright (2005). The derivative is its implication: closing the wedge is not optimal here. With the direct effects switched off, only the seller mass carries the policy, and a higher μ raises H^{B*} , which through the sunk cost d forces a larger surplus per trade. This free-entry channel outweighs the wedge at every admissible μ , so welfare rises toward the boundary $\bar{\mu}^B$ and falls toward the Friedman rule, which is the worst admissible policy; at the rule itself the equilibrium is degenerate, with no sellers and no trade.

Construction details. Two features complete the construction. First, for $\mu > 0$ the transfer rule is discontinuous at the threshold, and the value function inherits a pinned jump. An agent at the cutoff who remains a seller collects the transfer in every future period, while a buyer forfeits eligibility upon trading, so the buyer branch must sit above the seller branch at the cutoff by exactly the discounted value of the forfeited transfers, an eligibility rent proportional to μ . The jump $\Delta^v > 0$ is thus forced and pinned by the policy discontinuity, in contrast to the drop, where Proposition 3 excludes any jump. Second, for $\mu < 0$ the rent reverses sign, the equilibrium selects the continuous member, and three devices are required: the stationary buyer distribution is the point mass at $\bar{m}_b$, since deviations from $\bar{m}_b$ grow without bound under the no-trade map; the eligibility threshold sits strictly inside the buyer region, since at the cutoff itself a seller could shave its receipt and escape the tax, leaving the Nash program without an optimal payment; and production below the cutoff is mandatory, since $v^*(0) < 0$ under deflation and free role choice would fail. The welfare objects are selection-free, and (A.4) holds on the whole range under these devices.

B.1 Proofs for the Baseline Equilibrium

This appendix proves Lemma 2 and Proposition 2. The proof first derives the value-function family and then verifies bargaining, role choice, and stationarity.

B.1.1 Proof of Lemma 2

Proof. Step 1: Baseline identities. We first collect the equations from Section 3 used in the construction.

Environment: as in Section 2, with measure-one continuum, discount $\beta \in (0, 1)$, utility $u^b(x) = x$, cost $u^s(x) = -cx$ with $c \in (0, 1)$, money stock $M > 0$, role choice $f(m) \in \{B, S\}$ each period, sunk cost $d > 0$ paid by S -agents before matching, matching probability $\alpha \in (0, 1)$, Nash bargaining with buyer power $\theta \in (0, 1)$. Pay-all equilibrium (the main text's Definition 2): (i) trades bind at $p = m_b$; (ii) $f(m) = S$ iff $m < \tilde{m}$. The main equations of Sections 2–3:

On-path bargaining. With $p(m) = m$ and $x(m) > 0$,

$$cx(m) = \beta[\theta + (1 - \theta)c](v(m) - v(0)), \quad (\text{B.1})$$

and the joint surplus $J(m) = (1 - c)x(m)$ is split as

$$J^b(m) = \tilde{\theta}J(m), \quad J^s(m) = (1 - \tilde{\theta})J(m), \quad \tilde{\theta} = \frac{\theta}{\theta + (1 - \theta)c}. \quad (\text{B.2})$$

On-path Bellman equations.

$$v(m) = \alpha H(0) \tilde{\theta} J(m) + \beta v(m), \quad m \in [\tilde{m}, \infty), \quad (\text{B.3})$$

$$v(0) = -d + \alpha \int_{\tilde{m}}^{\infty} (1 - \tilde{\theta}) J(m) dH(m) + \beta v(0). \quad (\text{B.4})$$

Lemma 1 gives

$$H^*(0) = \frac{(1 - \beta)c}{\alpha\beta\theta(1 - c)}, \quad v^*(0) = 0, \quad (\text{B.5})$$

with $H^*(0) \in (0, 1)$ if and only if $\alpha\beta\theta(1 - c) > (1 - \beta)c$. We use (B.5) throughout.

Intuition. Money is an asset whose only dividend is trading surplus. At the margin, the buyer Bellman equation (B.3) says each unit of money must earn its required return through the extra surplus it buys,

$$(1 - \beta) v'(m) = \alpha H^*(0) \cdot \tilde{\theta} J'(m),$$

By the bargaining condition (B.1), marginal surplus is proportional to the marginal value of money:

$$J'(m) = \frac{(1 - c)[\theta + (1 - \theta)c]\beta}{c} v'(m).$$

The zero-money mass is the market tightness that delivers exactly the required return: with scarcer sellers money would under-earn, with more abundant sellers it would over-earn. Cancelling $v'(m)$ gives the form used repeatedly below:

$$\frac{\alpha H^*(0) \theta \beta (1 - c)}{c} = 1 - \beta. \quad (\text{B.6})$$

Note that $v'(m)$ cancels: the condition is the same on both branches of v . And $v^*(0) = 0$ says the zero-balance position itself carries no rent: all value is carried by money.

Distributional restrictions. $H(0) = H^*(0)$ and $H((0, \tilde{m})) = 0$, with $\int_{\tilde{m}}^{\infty} dH = 1 - H^*(0)$ and $\int_{\tilde{m}}^{\infty} m dH = M$; buyers' average balance is $\bar{m}_b \equiv M/(1 - H^*(0))$.

Aggregates. With $Z \equiv \alpha \beta (1 - c)(1 - \theta)(1 - H^*(0))$, Recall that $K \equiv c + \theta - c\theta$, so $\tilde{\theta} = \theta/K$ and $1 - \tilde{\theta} = c(1 - \theta)/K$.

$$\bar{J}^* = \frac{d}{\alpha(1 - H^*(0))(1 - \tilde{\theta})}, \quad \bar{v}_b^* = \frac{\alpha H^*(0) \tilde{\theta}}{1 - \beta} \bar{J}^* = \frac{d}{Z}, \quad \bar{x}^* = \frac{\bar{J}^*}{1 - c}, \quad (\text{B.7})$$

where $\bar{J}^*$ and $\bar{v}_b^*$ are the buyer-population averages of $J(m)$ and $v(m)$. Derivation: averaging (B.3) over the buyer distribution gives $(1 - \beta) \bar{v}_b^* = \alpha H^*(0) \tilde{\theta} \bar{J}^*$; the zero-balance Bellman equation (B.4) with $v(0) = 0$ gives $d = \alpha(1 - \tilde{\theta}) \int_{\tilde{m}}^{\infty} J dH = \alpha(1 - \tilde{\theta})(1 - H^*(0)) \bar{J}^*$, which is the first formula. For the second, substitute $1 - \tilde{\theta} = (1 - \theta)c/K$ into the first to get $\bar{J}^* =$

$Kd/[\alpha(1-H^*(0))(1-\theta)c]$, then use $\tilde{\theta} = \theta/K$ and (B.6) in the form $1-\beta = \alpha H^*(0)\theta\beta(1-c)/c$:

$$\begin{aligned}\bar{v}_b^* &= \frac{\alpha H^*(0) (\theta/K)}{\alpha H^*(0)\theta\beta(1-c)/c} \bar{J}^* = \frac{c}{K\beta(1-c)} \cdot \frac{Kd}{\alpha(1-H^*(0))(1-\theta)c} \\ &= \frac{d}{\alpha\beta(1-c)(1-\theta)(1-H^*(0))} = \frac{d}{Z}.\end{aligned}$$

These follow from (B.3)–(B.4) and (B.5) alone; nothing in this appendix changes them.

Step 2: The candidate.

Fix a cutoff $\tilde{m} > 0$ and a jump $\Delta^v \geq 0$, and conjecture a continuation value of the two-branch linear form:

$$v(m) = \begin{cases} v_0 + a m, & m < \tilde{m}, \\ v_0 + a \tilde{m} + \Delta^v + b(m - \tilde{m}), & m \geq \tilde{m}, \end{cases} \quad 0 < a < b, \quad (\text{B.8})$$

with (v_0, a, b, Δ^v) and $H(0)$ to be determined. Steps 3–4 characterize bargaining and the two role values. The proof of Proposition 2 then verifies that the candidate is their upper envelope, with the role choice switching at $\tilde{m}$. A jump does not contradict the envelope property: the buyer value inherits the discontinuity of v exactly, because a deviating buyer's payment can push a zero-balance partner across the cutoff and capture a share of the jump.

Step 3: Bargaining outcomes.

Consider a meeting in which a consumer holding m_b buys from a producer holding m_s , with payment $p \in [0, m_b]$ and quantity x . Write

$$\Delta_s(p) \equiv v(m_s + p) - v(m_s), \quad \Delta_b(p) \equiv v(m_b) - v(m_b - p),$$

for the two money-value increments, and define the *surplus index*

$$S(m_b, m_s, p) \equiv \beta \Delta_s(p) - c \beta \Delta_b(p). \quad (\text{B.9})$$

Both Δ 's, hence S , involve only *differences* of v , so the anchor v_0 is irrelevant throughout this section; we display all formulas at $v_0 = 0$.

Reduction to the index. The surpluses are $J^b = x - \beta\Delta_b$ and $J^s = \beta\Delta_s - cx$, and the interior Nash first-order condition in x is $\theta J^s = (1 - \theta)cJ^b$. Substituting $\beta\Delta_b = x - J^b$ and $\beta\Delta_s = J^s + cx$ into (B.9),

$$S = (J^s + cx) - c(x - J^b) = J^s + cJ^b,$$

and the first-order condition $J^s = (1 - \theta)cJ^b/\theta$ then gives

$$J^b = \frac{\theta}{c} S, \quad J^s = (1 - \theta) S, \quad cx = (1 - \theta)c\beta\Delta_b + \theta\beta\Delta_s, \quad (\text{B.10})$$

the last from $x = \beta\Delta_b + J^b$. Consequently the Nash product itself is *linear* in the index,

$$N \equiv (J^b)^\theta (J^s)^{1-\theta} = \left(\frac{\theta}{c}\right)^\theta (1 - \theta)^{1-\theta} S, \quad (\text{B.11})$$

so: (i) trade with interior x occurs iff $S > 0$, with both surpluses then positive; (ii) the bargained payment maximizes $S(p)$ over $p \in [0, m_b]$; (iii) any corner with $x \leq 0$ has $J^b = x - \beta\Delta_b \leq 0$ (as v is increasing, $\Delta_b \geq 0$), hence a nonpositive Nash product, and is dominated whenever some p attains $S > 0$. Finally, because v has the two-branch linear form, with slope a below $\tilde{m}$ and slope b above it, $S(\cdot)$ is piecewise linear in p with slope

$$S'(p) = \beta[v'(m_s + p) - cv'(m_b - p)], \quad v' \in \{a, b\}, \quad (\text{B.12})$$

stepping by $+\beta\Delta^v$ at a payment where the producer's post-trade balance reaches $\tilde{m}$ and by $-c\beta\Delta^v$ where the consumer's retained balance falls below it.

We write $S(m_b, m_s) \equiv S(m_b, m_s, p(m_b, m_s))$ for the index evaluated at the bargained payment. We now show that $p(m_b, m_s) = m_b$ and compute $S(m_b, m_s)$ in closed form.

Under the guess (B.8) with $0 < a < b$ (any v_0) and

$$0 \leq \Delta^v \leq \frac{(b - ca)\tilde{m}}{c}, \quad (\text{B.13})$$

in every meeting between a seller $m_s \in [0, \tilde{m})$ and a buyer $m_b \in [\tilde{m}, \infty)$ the bargaining outcome is $p(m_b, m_s) = m_b$, with

$$S(m_b, m_s) = (1 - c)\beta v(m_b) + \beta(b - a)m_s \geq (1 - c)\beta(a\tilde{m} + \Delta^v) > 0 \quad (\text{B.14})$$

(displayed at $v_0 = 0$), so $x > 0$, $J^b > 0$, $J^s > 0$. Every match of an off-role agent (a buyer with $m \in (0, \tilde{m})$ against a zero-balance seller, or a seller with $m \geq \tilde{m}$ against an on-path buyer) also trades at pay-all with positive surpluses; the single exception is a deviating buyer with $m = 0$, for whom $S \equiv 0$ and no trade occurs. At the upper bound of (B.13) pay-all ties with the corner $p = m_b - \tilde{m}$; strictly inside, it is the unique maximizer. This argument uses only differences of an arbitrary two-branch continuation profile and does not use stationarity or the Bellman equations. It therefore applies date by date in Section 6, where no balance transformation intervenes between trade and continuation.

Value at pay-all. The buyer retains 0 and the seller ends at $m_s + m_b \geq \tilde{m}$, so $\Delta_b = v(m_b) - v(0)$ and $\Delta_s = v(m_s + m_b) - v(m_s) = \Delta_b + (b - a)m_s$: the jump enters both increments once and cancels in their difference. Then $S = \beta\Delta_s - c\beta\Delta_b$ gives (B.14), and $v(m_b) \geq v(\tilde{m}) = a\tilde{m} + \Delta^v$.

Optimality of pay-all. By (B.11), compare $S(p)$ across the four regions determined by the positions of $m_s + p$ and $m_b - p$ relative to the cutoff.

Region (a): $m_s + p \geq \tilde{m}$, $m_b - p < \tilde{m}$ (contains $p = m_b$). By (B.12) the slope is $\beta(b - ca) > 0$: the corner $p = m_b$ wins.

Region (b): $m_s + p \geq \tilde{m}$ and $m_b - p \geq \tilde{m}$, so $p \leq m_b - \tilde{m}$. Here $\Delta_s = a(\tilde{m} - m_s) + \Delta^v + b(m_s + p - \tilde{m})$ and $\Delta_b = bp$, so S_b is increasing in p with supremum at $p = m_b - \tilde{m}$, and

$$S(m_b, m_s) - \sup_{p \leq m_b - \tilde{m}} S_b(p) = \beta(b - ca)\tilde{m} - c\beta\Delta^v \geq 0$$

by the upper bound in (B.13), with equality only at the bound.

Region (c): $m_s + p < \tilde{m}$ and $m_b - p < \tilde{m}$, so $m_b - \tilde{m} < p < \tilde{m} - m_s$. Here $\Delta_s = ap$ and $\Delta_b = (b - a)(m_b - \tilde{m}) + \Delta^v + ap$, so

$$\begin{aligned} S(m_b, m_s) - S_c(p) &= (1 - c)\beta a(\tilde{m} - p) + [(1 - c)b + c(b - a)]\beta(m_b - \tilde{m}) \\ &\quad + \beta(b - a)m_s + \beta\Delta^v \\ &\geq \beta\Delta^v \geq 0, \end{aligned}$$

with strict inequality at every feasible p ; the lower bound in (B.13) is exactly what keeps this region dominated in the limit $m_s \downarrow 0$, $p \uparrow \tilde{m}$, $m_b \downarrow \tilde{m}$.

Region (d): $m_s + p < \tilde{m} \leq m_b - p$. Here $\Delta_s = ap$, $\Delta_b = bp$, so $S_d(p) = \beta(a - cb)p$ with $p \leq$

$m_b - \tilde{m}$. If $a \leq cb$ then $S_d \leq 0 < S(m_b, m_s)$. If $a > cb$, the difference $S(m_b, m_s) - \beta(a - cb)p$ at its minimum $p = m_b - \tilde{m}$ has, as a function of m_b , slope $\beta[(1 - c)b - (a - cb)] = \beta(b - a) > 0$ and value $(1 - c)\beta(a\tilde{m} + \Delta^v) + \beta(b - a)m_s > 0$ at $m_b = \tilde{m}$: dominated.

A corner with $x \leq 0$ is dominated as in (B.11)(iii).

Off-role buyer $m \in (0, \tilde{m})$ against a zero-balance seller. Every payment $p \leq m$ leaves both parties below the cutoff: a single region with slope $(1 - c)\beta a > 0$, so pay-all; at $p = m$, $\Delta_b = \Delta_s = v(m) = am$ and $S = (1 - c)\beta am > 0$ for $m > 0$; at $m = 0$, $S \equiv 0$ and no trade occurs.

Off-role seller $m \geq \tilde{m}$ against a buyer $m_b \geq \tilde{m}$. Both of the seller's positions stay on the upper branch, so at pay-all $\Delta_s = b m_b$ carries no jump while $\Delta_b = v(m_b)$ keeps it:

$$\begin{aligned} S(m_b, m) &= (1 - c)\beta b m_b + c\beta[(b - a)\tilde{m} - \Delta^v] \\ &\geq (1 - c)\beta b (m_b - \tilde{m}) \geq 0, \end{aligned}$$

strictly for $m_b > \tilde{m}$, by (B.13); the single downward step of S , at $p = m_b - \tilde{m}$, is $-c\beta\Delta^v$, and the corner comparison is the same as in region (b), so pay-all remains attained.

Step 4: Role values and the value-function family.

Given the candidate and the bargaining outcomes of Step 3, define for every $m \geq 0$ the two role values (all on-path sellers hold 0; displayed at $v_0 = 0$):

$$v_S(m) = -d + \alpha(1 - H(0))(1 - \theta) [S(\bar{m}_b, m)]^+ + \beta v(m), \quad (\text{B.15})$$

$$v_B(m) = \alpha H(0) \frac{\theta}{c} [S(m, 0)]^+ + \beta v(m), \quad (\text{B.16})$$

where the buyer-side average in (B.15) is exact at the mean $\bar{m}_b = M/(1 - H(0))$ because $S(m_b, m)$ is affine in m_b and positive.

Assume (30) and $d > 0$. The identities $v_S(m) = v(m)$ on $[0, \tilde{m})$ and $v_B(m) = v(m)$ on $[\tilde{m}, \infty)$ hold if and only if $H(0) = H^*(0)$, $v_0 = 0$,

$$a = \chi b, \quad \chi = \frac{s}{1 - \beta + s}, \quad s = \alpha\beta(1 - \theta)(1 - H^*(0)), \quad (\text{B.17})$$

and the level condition $Z[\Delta^v + a\tilde{m} + b(\bar{m}_b - \tilde{m})] = d$, which, with $a = \chi b$ from (B.17), solves

to

$$b = b(\tilde{m}, \Delta^v) = \frac{d/Z - \Delta^v}{\bar{m}_b - (1 - \chi)\tilde{m}} > 0. \quad (\text{B.18})$$

For each pair $(\tilde{m}, \Delta^v)$ with $\tilde{m} \in (0, \bar{m}_b)$ and $0 \leq \Delta^v < d/Z$ the candidate is uniquely determined, and every member passes through the average buyer's pair $(\bar{m}_b, d/Z)$. Moreover, on all of $[0, \infty)$,

$$v_B(m) = v(m), \quad v(m) - v_S(m) = \begin{cases} 0, & m \in [0, \tilde{m}), \\ (1 - \beta)b(m - \tilde{m}) + (1 - \beta + s)\Delta^v, & m \in [\tilde{m}, \infty). \end{cases} \quad (\text{B.19})$$

Substep (a): the buyer value equals the value function. Against a zero-balance partner, pay-all gives $\Delta_b = \Delta_s = v(m) - v(0) = v(m)$ on both branches by (B.14), so $S(m, 0) = (1 - c)\beta v(m)$: the deviating buyer's harvest inherits the jump of v , because at $m \geq \tilde{m}$ the payment pushes the partner across the cutoff. By (B.6), $\alpha H^*(0)\theta\beta(1 - c)/c = 1 - \beta$, so $v_B(m) = (1 - \beta)v(m) + \beta v(m) = v(m)$ for every m ; conversely, matching the m -coefficient of v_B with b on the upper branch forces $H(0) = H^*(0)$, and $v_B(0) = v(0)$ forces $\beta v_0 = v_0$, i.e. $v_0 = 0$.

Substep (b): the seller value below the cutoff. For $m < \tilde{m}$, (B.14) gives $S(\bar{m}_b, m) = (1 - c)\beta[\Delta^v + a\tilde{m} + b(\bar{m}_b - \tilde{m})] + \beta(b - a)m$, affine in m . Matching the m -coefficient of (B.15) with a gives $\alpha(1 - H^*(0))(1 - \theta)\beta(b - a) + \beta a = a$, which is (B.17); matching the level at $m = 0$ gives $0 = -d + Z[\Delta^v + a\tilde{m} + b(\bar{m}_b - \tilde{m})]$, the level condition, and (B.18) follows by substitution.

Substep (c): the seller value at and above the cutoff. For $m \geq \tilde{m}$ the seller's two positions lie on the upper branch, so $\Delta_s = b\bar{m}_b$ carries no jump while $\Delta_b = v(\bar{m}_b)$ keeps it, and $S(\bar{m}_b, m) = \beta b\bar{m}_b - c\beta v(\bar{m}_b)$, independent of m . Using the level condition and $s = \alpha\beta(1 - \theta)(1 - H^*(0))$,

$$v_S(m) = \beta v(m) - s[\Delta^v - (b - a)\tilde{m}] \quad \text{on } [\tilde{m}, \infty),$$

so $v(m) - v_S(m) = (1 - \beta)v(m) + s[\Delta^v - (b - a)\tilde{m}]$; substituting $v(m) = a\tilde{m} + \Delta^v + b(m - \tilde{m})$, the $\tilde{m}$ -terms cancel by (B.17), and the second line of (B.19) remains. $\square$

B.1.2 Proof of Proposition 2

Proof. Step 1: Role optimality across the cutoff.

For the family characterized by (B.17)–(B.19), on $[0, \tilde{m})$ the role choice is exactly indifferent, $v_B = v_S = v$; at and above $\tilde{m}$ the buyer role is optimal with the gap in (B.19), strict for $m > \tilde{m}$ and equal to $(1 - \beta + s)\Delta^v \geq 0$ at $m = \tilde{m}$. If instead $\Delta^v < 0$, the candidate fails both role optimality at the cutoff and Nash bargaining for buyers just above it. Downward jumps are therefore excluded by equilibrium; upward jumps survive up to the pay-all bound (B.13).

The first claims restate (B.19), whose derivation nowhere used the sign of Δ^v . For $\Delta^v < 0$, evaluating (B.19) at $m = \tilde{m}$ gives $v(\tilde{m}) - v_S(\tilde{m}) = (1 - \beta + s)\Delta^v < 0$; and in region (a) of the pay-all comparison in the proof of Lemma 2, the step of S at the payment $p = \tilde{m} - m_s$ becomes $\beta\Delta^v < 0$, so S is maximized only in the limit $p \uparrow \tilde{m} - m_s$ whenever the slope gain $\beta(b - ca)(m_b - \tilde{m} + m_s)$ on the remaining stretch falls short of the step, which is the stated range at $m_s = 0$.

Interpretation. Why the freedom is two-dimensional: by (B.6), free entry equalizes the role levels at $m = 0$, and a buyer's surplus share exactly offsets the flow cost of holding balances at every m , so $v_B = v$ identically and role choice below the cutoff is indifferent. Neither the cutoff nor the jump is pinned; the profile is disciplined only by $\Delta^v \geq 0$ and the pay-all bound (B.13). A jump is compatible with the upper envelope because the envelope itself jumps: the deviating buyer's harvest $S(m, 0) = (1 - c)\beta v(m)$ is discontinuous at $\tilde{m}$.

Step 2: Admissible jumps and completion. Assume (30) and $d > 0$. For every $\tilde{m} \in (0, \bar{m}_b)$ and every $\Delta^v \in [0, \bar{\Delta}(\tilde{m})]$, where

$$\bar{\Delta}(\tilde{m}) = \frac{d}{Z} \cdot \frac{(1 - c\chi) \tilde{m}}{c(\bar{m}_b - (1 - \chi)\tilde{m}) + (1 - c\chi) \tilde{m}} \quad (\text{B.20})$$

is the pay-all bound (B.13) evaluated at (B.17) and (B.18), the candidate characterized by (B.17)–(B.19), together with any distribution satisfying (3)–(4), is a stationary pay-all equilibrium: $v = \max\{v_S, v_B\}$ at every $m \geq 0$ with the assigned roles attaining the maximum, all meetings trade at pay-all with positive surpluses, and the flows reproduce the distribution. Equations (B.15)–(B.19) and Step 1 give the Bellman equations and role optimality: $v_B = v$ everywhere and $v_S = v$ on the seller region, so v is the upper envelope

and the assigned role attains it, with indifference on $[0, \tilde{m})$ and at $m = \tilde{m}$ when $\Delta^v = 0$. The pay-all comparison in the proof of Lemma 2, together with (B.13)–(B.14), verifies bargaining. Stationarity is unchanged from the continuous construction: a matched buyer pays all and re-enters at 0 as a seller; a matched seller receives $m_b \geq \tilde{m}$ and crosses into the buyer region; unmatched agents stay put, and the restrictions (3)–(4) are preserved. $\square$

B.2 Proofs for the Policy in Section 5

This appendix proves Lemmas 3–4 and Proposition 3. The proofs first characterize the stationary distribution and then verify the candidate family, bargaining, role choice, stationarity, and uniqueness.

B.2.1 Proof of Lemma 3

Proof. Throughout, $\mu > 0$, the trading pattern is stationary pay-all, and the branch-preservation condition in Definition 3 holds; hence the cutoff is M . Fix the seller mass $H \in (0, 1)$.

Step 1: Policy drift and the cutoff.

Every agent now receives the lump-sum transfer μM_{t-1}^n at the start of each period ($\mu > 0$; M_{t-1}^n is the nominal stock). In real terms a balance m moves, absent trade, to

$$g(m) = \frac{m + \mu M}{1 + \mu}, \tag{B.21}$$

whose unique fixed point is M : balances above M drift down, balances below drift up. Note for later use that g is increasing, that $g(y) < M \iff y < M$, and that $g(y') - g(y) = (y' - y)/(1 + \mu)$; also the real stock is conserved, $\int g d\Phi = (M + \mu M)/(1 + \mu) = M$ for any distribution Φ with mean M .

Clause 2 of Definition 3 forces the cutoff to M . It requires the role regions to be *invariant under the no-trade drift*: an agent who does not trade keeps the same role. This is a restriction, not an implication of the Bellman equations: without it, an unmatched agent could drift across the cutoff and switch roles, and the stationary flows would have to be re-solved for the resulting migration, which we do not pursue. Within this class the cutoff is pinned. Since not trading for t consecutive periods has positive probability and moves a

balance to $g^t(m) \rightarrow M$ monotonically, invariance requires $g(m) \geq \tilde{m} \iff m \geq \tilde{m}$. Solving, $g(m) \geq \tilde{m} \iff m \geq \tilde{m} + \mu(\tilde{m} - M)$, which coincides with $m \geq \tilde{m}$ iff $\tilde{m} = M$. Concretely: if $\tilde{m} < M$, a seller with m just below $\tilde{m}$ drifts across the cutoff without ever trading and becomes a buyer; if $\tilde{m} > M$, a buyer with m slightly above $\tilde{m}$ drifts down across it. Under clause 2 the policy thus eliminates the cutoff freedom of the $\mu = 0$ model; Proposition 3 shows that the role-choice conditions then also eliminate the jump and the slope freedom.

Step 2: Seller grid, masses, and means. Fix a seller mass $H \in (0, 1)$ and the pay-all trading pattern with cutoff M . In the stationary distribution, sellers occupy the grid

$$\ell_n = \left[1 - (1 + \mu)^{-n}\right]M, \quad n = 1, 2, \dots, \quad (\text{B.22})$$

with population masses and within-seller probabilities

$$\omega_n = \alpha H (1 - H) \rho^{n-1}, \quad w_n = \frac{\omega_n}{H} = \alpha (1 - H) \rho^{n-1}, \quad \rho = 1 - \alpha (1 - H), \quad (\text{B.23})$$

and the conditional means are

$$\bar{m}_s \equiv \sum_{n \geq 1} w_n \ell_n = \frac{\mu M}{\mu + \alpha(1 - H)}, \quad \bar{m}_b = \frac{(\mu + \alpha)M}{\mu + \alpha(1 - H)}. \quad (\text{B.24})$$

These objects are payoff-independent. (The buyer-side distribution beyond its mean is not needed below: every seller-side equation is affine in the partner's balance.) *The grid.* A buyer who trades pays all and ends the period at 0; at the start of the next period the transfer moves it to $g(0) = \mu M / (1 + \mu) = \ell_1$, and it is a seller ($\ell_1 < M$). An unmatched seller at ℓ_n drifts to

$$g(\ell_n) = \frac{[1 - (1 + \mu)^{-n}]M + \mu M}{1 + \mu} = \frac{[(1 + \mu) - (1 + \mu)^{-n}]M}{1 + \mu} = \left[1 - (1 + \mu)^{-(n+1)}\right]M = \ell_{n+1},$$

and $\ell_n \uparrow M$: a fresh spender walks the grid $0 \rightarrow \ell_1 \rightarrow \ell_2 \rightarrow \dots$, never reaching M .

The masses. Each period a seller meets a buyer with probability $\alpha(1 - H)$; upon trading it receives $m_b \geq M$ and leaves the seller region. Otherwise it drifts one grid step. Hence stationarity requires $\omega_{n+1} = \rho \omega_n$ with $\rho = 1 - \alpha(1 - H)$. The inflow to ℓ_1 is the mass of buyers who traded in the previous period: buyers have mass $1 - H$ and each meets a seller

with probability αH , so $\omega_1 = \alpha H(1 - H)$. Consistency: the total seller mass is

$$\sum_{n \geq 1} \omega_n = \frac{\alpha H(1 - H)}{1 - \rho} = \frac{\alpha H(1 - H)}{\alpha(1 - H)} = H,$$

and the outflow of sellers, $\alpha(1 - H) \sum_n \omega_n = \alpha H(1 - H)$, equals the inflow, as required. The within-seller probabilities are $w_n = \omega_n / H$, with $\sum_n w_n = 1$.

The means. Using $\sum_{n \geq 1} \rho^{n-1} (1 + \mu)^{-n} = \frac{1}{(1+\mu)-\rho} = \frac{1}{\mu + \alpha(1-H)}$,

$$\bar{m}_s = M \sum_n w_n \left[1 - (1 + \mu)^{-n} \right] = M \left[1 - \frac{\alpha(1 - H)}{\mu + \alpha(1 - H)} \right] = \frac{\mu M}{\mu + \alpha(1 - H)}.$$

For $\bar{m}_b$, the real stock is conserved at M , so $H\bar{m}_s + (1 - H)\bar{m}_b = M$ and

$$\bar{m}_b = \frac{M - H\bar{m}_s}{1 - H} = \frac{M[\mu + \alpha(1 - H)] - H\mu M}{(1 - H)[\mu + \alpha(1 - H)]} = \frac{M(1 - H)(\mu + \alpha)}{(1 - H)[\mu + \alpha(1 - H)]} = \frac{(\mu + \alpha)M}{\mu + \alpha(1 - H)}.$$

The buyer side need not be solved in closed form: one transition operator gives all the properties needed in Lemma 3.

Step 3: Buyer distribution.

Fix $H \in (0, 1)$ as in Step 2 and write a buyer's excess balance as $Y = m - M$. Let I be independent of Y and the seller draw ℓ_J , with $\Pr(I = 1) = \alpha H$ and $\Pr(J = n) = w_n$. For a law Φ of Y , let $T(\Phi)$ be the distribution of $(Y + I\ell_J)/(1 + \mu)$ when $Y \sim \Phi$. When $I = 0$, an unmatched buyer remains in the buyer pool. When $I = 1$, the buyer trades and leaves, while the matched seller inherits its balance and enters the buyer pool with excess $(Y + \ell_J)/(1 + \mu)$. Thus $T(\Phi)$ is the one-period law of buyer excess balances.

The interval $[0, M/\mu]$ is invariant under T : if $Y \in [0, M/\mu]$ and $\ell_J < M$, then $(Y + I\ell_J)/(1 + \mu) \in [0, M/\mu]$. For two laws Φ and $\tilde{\Phi}$ on this interval, the 1-Wasserstein distance is $W_1(\Phi, \tilde{\Phi}) \equiv \inf \mathbb{E} |Y - \tilde{Y}|$, where the infimum is over all couplings of Φ and $\tilde{\Phi}$: joint draws $(Y, \tilde{Y})$ whose marginal laws are Φ and $\tilde{\Phi}$. Given any coupling, drawing the same I and ℓ_J for both populations couples $T(\Phi)$ and $T(\tilde{\Phi})$ with gap $|Y - \tilde{Y}|/(1 + \mu)$. Taking expectations and then the infimum over couplings gives

$$W_1(T(\Phi), T(\tilde{\Phi})) \leq \frac{1}{1 + \mu} W_1(\Phi, \tilde{\Phi}). \quad (\text{B.25})$$

The space of probability laws on the compact invariant interval is complete under W_1 , so the contraction mapping theorem gives a unique fixed point Φ^* . Since $m = M + Y$, buyer balances are supported on $[M, (1 + \mu)M/\mu]$.

To obtain the CDF equation, let $\hat{B}(y) \equiv B^*(y)/(1 - H)$ be the conditional CDF of the buyer balance, extended by zero and one outside its support. The fixed-point condition $T(\Phi^*) = \Phi^*$ gives

$$\hat{B}(y) = (1 - \alpha H)\hat{B}((1 + \mu)y - \mu M) + \alpha H \sum_{j \geq 1} w_j \hat{B}((1 + \mu)y - \mu M - \ell_j). \quad (\text{B.26})$$

Using $w_j = \omega_j/H$, multiplying by $1 - H$, and replacing the sum over seller atoms by integration against H^* gives (33). Hence the unique fixed point is also the unique solution of that CDF equation.

Finally, taking expectations in $T(\Phi^*) = \Phi^*$ gives

$$\mathbb{E}[Y] = \frac{\mathbb{E}[Y] + \alpha H \bar{m}_s}{1 + \mu} \implies \mu \mathbb{E}[Y] = \alpha H \bar{m}_s. \quad (\text{B.27})$$

Therefore $\mathbb{E}[m] = M + \alpha H \bar{m}_s/\mu = \bar{m}_b$ by (B.24).

□

B.2.2 Proof of Lemma 4

Proof. Throughout, $\mu > 0$, $H^*(L) \in (0, 1)$, and the candidate is a stationary two-branch linear pay-all profile with cutoff M . The Bellman equations (38)–(39) and the Nash surplus split therefore apply.

Write $H \equiv H^*(L)$ and $s \equiv \alpha\beta(1 - \theta)(1 - H)$. The jump and the level of the value function do not enter the coefficient equations.

Step 1: Seller mass and slope ratio. For a buyer with $m \geq M$, matching the coefficient on m in the buyer Bellman equation gives

$$b = \left[\frac{\alpha H \theta \beta (1 - c)}{c} + \beta \right] \frac{b}{1 + \mu}.$$

Since $b > 0$, this is equivalent to

$$\frac{\alpha H \theta \beta}{c} = \frac{1 - \beta + \mu}{1 - c}, \quad (\text{B.28})$$

and hence to (40). For a seller with $m < M$, coefficient matching similarly gives

$$a = \frac{s(b - a) + \beta a}{1 + \mu}, \quad \text{or equivalently} \quad a(1 - \beta + \mu) = s(b - a).$$

Thus $a = \chi b$, with $\chi = s/(1 - \beta + \mu + s)$ as in (41). Because $H \in (0, 1)$, we have $s > 0$ and therefore $\chi \in (0, 1)$. □

B.2.3 Proof of Proposition 3

Proof. Under the hypotheses of Proposition 3, the conditions of Lemmas 3 and 4 are satisfied. We may therefore use the stationary distribution, the conditional means (34), and the coefficient identities (40)–(41) below.

Step 1: Bellman constants and the candidate family.

For a candidate of the form (35), write $H \equiv H^*(L)$, $v_0 \equiv v(0)$, and use $a = \chi b$ from Lemma 4. Let $Z(\mu) \equiv \alpha\beta(1 - c)(1 - \theta)(1 - H)$. Evaluating the buyer and seller Bellman equations at their branch anchors and collecting constant terms gives

$$(1 - \beta)v_0 = \mu\Delta^v + \frac{\beta a \mu M}{1 + \mu} + \frac{\alpha\beta\theta(b - a)M^*}{c(1 + \mu)}, \quad (\text{B.29})$$

$$(1 - \beta)v_0 = -d + Z(\mu)\Delta^v + \frac{\alpha\beta(1 - c)(1 - \theta)b(M - M^*) - Z(\mu)(b - a)M + \beta a \mu M}{1 + \mu}. \quad (\text{B.30})$$

Subtracting (B.30) from (B.29), using $a = \chi b$ and the conditional means in (34), gives

$$L(\mu)b = d - (Z(\mu) - \mu)\Delta^v. \quad (\text{B.31})$$

Either (B.29) or (B.30) then determines v_0 . Equation (B.31) is one affine restriction on (b, Δ^v) . If $Z(\mu) \neq \mu$, b indexes the family; if $L(\mu) \neq 0$, Δ^v indexes it. At $Z(\mu) = \mu$, the equation pins b and leaves the jump free; at $L(\mu) = 0$, it pins the jump and leaves the slope

free. If both coefficients vanish, (B.31) would read $0 = d$, so no candidate exists.

At the continuous member $\Delta^v = 0$, the same two constant-term equations can be written as

$$(1 - \beta)(v_0 + aM) = (1 - \beta + \mu) \frac{aM + (b - a)\overline{m}_s/(1 - c)}{1 + \mu}, \quad (\text{B.32})$$

$$(1 - \beta)v_0 = -d + Z(\mu) \frac{b(\overline{m}_b - M) + aM}{1 + \mu} + \frac{\beta a \mu M}{1 + \mu}. \quad (\text{B.33})$$

Subtracting the second equation from the first yields $L(\mu)b = d$. Moreover, (B.28), $M^* = H\overline{m}_s$, and (B.29) give the equality in (44).

Step 2: Candidate and bargaining.

The cutoff is forced to M by the drift argument in the proof of Lemma 3; the candidate is a two-branch profile with an anchor and a free jump:

$$v(m) = \begin{cases} v_0 + a m, & m < M, \\ v_0 + a M + \Delta^v + b(m - M), & m \geq M, \end{cases} \quad 0 < a < b, \quad \Delta^v \in \mathbb{R}, \quad (\text{B.34})$$

with (v_0, a, b, Δ^v) and the seller mass H to be determined; $v_0 = v^*(0)$ is no longer zero. The surplus index now has *drifted* arguments: for a consumer m_b , a producer m_s , and a payment $p \in [0, m_b]$,

$$S(m_b, m_s, p) = \beta \Delta_s(p) - c\beta \Delta_b(p), \quad \begin{aligned} \Delta_s(p) &= v(g(m_s + p)) - v(g(m_s)), \\ \Delta_b(p) &= v(g(m_b)) - v(g(m_b - p)). \end{aligned} \quad (\text{B.35})$$

The reduction (B.10)–(B.11) in Appendix B.1.1 holds verbatim because it used only $J^b = x - \beta\Delta_b$, $J^s = \beta\Delta_s - cx$ and the Nash first-order condition, not the form of the Δ 's. Thus again $J^b = (\theta/c)S$, $J^s = (1 - \theta)S$, $cx = (1 - \theta)c\beta\Delta_b + \theta\beta\Delta_s$, the Nash product is linear in S , and the bargained payment maximizes $S(p)$. Because $g(y) < M \iff y < M$ and g contracts increments by $1/(1 + \mu)$, the index is piecewise linear in p with slope

$$S'(p) = \frac{\beta[v'(g(m_s + p)) - c v'(g(m_b - p))]}{1 + \mu}, \quad v' \in \{a, b\} : \quad (\text{B.36})$$

all slopes carry the common factor $1/(1 + \mu)$; under a nonzero jump the index also steps by

$+\beta\Delta^v$ at the payment where the producer's post-trade balance reaches M , and by $-c\beta\Delta^v$ at the payment where the consumer's retained balance falls below M . As before, $S(m_b, m_s) \equiv S(m_b, m_s, p(m_b, m_s))$ denotes the index at the bargained payment, and v_0 drops out of every formula in this section.

Under the guess (B.34) with $b > a > 0$ and

$$0 \leq \Delta^v \leq \frac{(b - ca)M}{c(1 + \mu)}, \quad (\text{B.37})$$

in every meeting between a seller $m_s \in [0, M)$ and a buyer $m_b \in [M, \infty)$ the bargaining outcome is $p(m_b, m_s) = m_b$, with

$$\begin{aligned} S(m_b, m_s) &= (1 - c)\beta \Delta_b(m_b) + \frac{\beta(b - a)m_s}{1 + \mu}, \\ \Delta_b(m_b) &= \frac{aM + b(m_b - M)}{1 + \mu} + \Delta^v \geq \frac{aM}{1 + \mu} + \Delta^v. \end{aligned} \quad (\text{B.38})$$

so $S \geq (1 - c)\beta[aM/(1 + \mu) + \Delta^v] > 0$, $x > 0$, $J^b > 0$, $J^s > 0$, with margins bounded away from zero uniformly as $\Delta^v \rightarrow 0$. Moreover every match of an off-role agent (a buyer with $m \in (0, M)$ against a grid partner, or a seller with $m \geq M$ against an on-path buyer) also trades at pay-all with positive surpluses; the single exception is a deviating buyer with $m = 0$, for whom $S \equiv 0$ and no trade occurs. Both bounds in (B.37) are used: the lower one keeps payments that leave the pair below the cutoff dominated, the upper one keeps retention above the cutoff dominated. *Value at pay-all.* At $p = m_b$ the buyer retains 0 and the seller holds $m_s + m_b \geq M$ (as $m_b \geq M$), so

$$\begin{aligned} \Delta_b(m_b) &= v(g(m_b)) - v(g(0)) = \left[v_0 + aM + \Delta^v + \frac{b(m_b - M)}{1 + \mu} \right] - \left[v_0 + \frac{a\mu M}{1 + \mu} \right] \\ &= \frac{aM + b(m_b - M)}{1 + \mu} + \Delta^v, \\ \Delta_s(m_b) &= v(g(m_s + m_b)) - v(g(m_s)) = \Delta_b(m_b) + \frac{(b - a)m_s}{1 + \mu}, \end{aligned} \quad (\text{B.39})$$

the jump entering both increments once and cancelling in their difference. Then $S = \beta\Delta_s - c\beta\Delta_b$ gives (B.38); $cx = (1 - \theta)c\beta\Delta_b + \theta\beta\Delta_s > 0$; and $J^b, J^s > 0$ from $S > 0$.

Optimality of pay-all. By linearity of the Nash product in S ((B.11)), compare $S(p)$ against (B.38) on the four regions (all displays carry the common factor $1/(1 + \mu)$, which we

keep explicit).

Region (a): $m_s + p \geq M$, $m_b - p < M$ (contains $p = m_b$). By (B.36) the slope is $\beta(b - ca)/(1 + \mu) > 0$ throughout the region: S is maximized at the corner $p = m_b$.

Region (b): $m_s + p \geq M$ and $m_b - p \geq M$, so $p \leq m_b - M$. Both post-trade balances are on the b -branch, only the seller crosses:

$$\Delta_s(p) = \frac{a(M - m_s) + b(m_s + p - M)}{1 + \mu} + \Delta^v, \quad \Delta_b(p) = \frac{bp}{1 + \mu},$$

so $S_b(p)$ is increasing in p with supremum at $p = m_b - M$, and

$$S(m_b, m_s) - \sup_{p \leq m_b - M} S_b(p) = \frac{\beta M(b - ca)}{1 + \mu} - c\beta\Delta^v \geq 0$$

by the upper bound in (B.37), with strict inequality when (B.37) is strict; at $\Delta^v = 0$ the margin is $\beta M(b - ca)/(1 + \mu) > 0$.

Region (c): $m_s + p < M$ and $m_b - p < M$, so $m_b - M < p < M - m_s$. Both post-trade balances are on the a -branch: $\Delta_s(p) = ap/(1 + \mu)$ and, the buyer's pre-trade position alone carrying the jump,

$$\Delta_b(p) = \frac{(b - a)(m_b - M) + ap}{1 + \mu} + \Delta^v,$$

so

$$\begin{aligned} S(m_b, m_s) - S_c(p) &= \frac{(1 - c)\beta[aM + b(m_b - M) - ap] + \beta(b - a)m_s}{1 + \mu} \\ &\quad + \frac{c\beta(b - a)(m_b - M)}{1 + \mu} + \beta\Delta^v \\ &> \beta\Delta^v \geq 0, \end{aligned}$$

since $p < M$ gives $aM - ap > 0$ and all remaining terms are nonnegative. This is the binding comparison: as $m_s \downarrow 0$, $p \uparrow M$, and $m_b \downarrow M$ its non-jump part vanishes, so the lower bound $\Delta^v \geq 0$ in (B.37) is exactly what keeps region (c) dominated.

Region (d): $m_s + p < M \leq m_b - p$. Here $\Delta_s(p) = ap/(1 + \mu)$, $\Delta_b(p) = bp/(1 + \mu)$, so

$S_d(p) = \beta(a - cb)p/(1 + \mu)$ with $p \leq m_b - M$. If $a \leq cb$ then $S_d \leq 0 < S(m_b, m_s)$. If $a > cb$, the supremum is at $p = m_b - M$ and

$$\begin{aligned}\psi(m_b) &\equiv S(m_b, m_s) - \frac{\beta(a - cb)(m_b - M)}{1 + \mu} \\ &= \frac{(1 - c)\beta[aM + b(m_b - M)] + \beta(b - a)m_s}{1 + \mu} \\ &\quad - \frac{\beta(a - cb)(m_b - M)}{1 + \mu} + (1 - c)\beta\Delta^v\end{aligned}$$

has m_b -coefficient $\beta[(1 - c)b - (a - cb)]/(1 + \mu) = \beta(b - a)/(1 + \mu) > 0$ and value $\psi(M) \geq [(1 - c)\beta aM + \beta(b - a)m_s]/(1 + \mu) > 0$, so $\psi > 0$ for all $m_b \geq M$.

A corner with $x \leq 0$ is dominated as in (B.11)(iii).

Off-role buyer $m \in (0, M)$ against a grid partner ℓ_j . For any $p \in [0, m]$ the buyer's retained balance satisfies $m - p < M$, so $v'(g(m - p)) = a$, and by (B.36) the slope of S is at least $\beta(a - ca)/(1 + \mu) = (1 - c)\beta a/(1 + \mu) > 0$ everywhere; at the payment where the partner crosses the cutoff, S steps *up* by $\beta\Delta^v$, so the maximum is still the corner: pay-all. At $p = m$,

$$\Delta_b = v(g(m)) - v(g(0)) = \frac{a(m + \mu M) - a\mu M}{1 + \mu} = \frac{am}{1 + \mu},$$

and for Δ_s there are two cases. If $\ell_j + m < M$, both of the partner's positions are on the a -branch and $\Delta_s = a[g(\ell_j + m) - g(\ell_j)] = am/(1 + \mu)$. If $\ell_j + m \geq M$,

$$\begin{aligned}\Delta_s &= \left[v_0 + aM + \Delta^v + \frac{b(\ell_j + m - M)}{1 + \mu} \right] - \left[v_0 + \frac{a(\ell_j + \mu M)}{1 + \mu} \right] \\ &= \frac{am + (b - a)(\ell_j + m - M)}{1 + \mu} + \Delta^v,\end{aligned}$$

using $a(M - \ell_j) = am - a(\ell_j + m - M)$. In both cases

$$\Delta_s = \frac{am + (b - a)(\ell_j + m - M)^+}{1 + \mu} + \Delta^v \mathbf{1}\{\ell_j + m \geq M\}, \quad (\text{B.40})$$

so $S = \beta[(1 - c)am + (b - a)(\ell_j + m - M)^+]/(1 + \mu) + \beta\Delta^v \mathbf{1}\{\ell_j + m \geq M\} \geq (1 - c)\beta am/(1 + \mu) > 0$ for $m > 0$, and $x > 0$; at $m = 0$, $S \equiv 0$ and no trade occurs.

Off-role seller $m \geq M$ against an on-path buyer $m_b \geq M$. For any p , $m + p \geq M$ so

$v'(g(m+p)) = b$, and the slope of S is positive on both linear pieces; the single step of S , at the payment $p = m_b - M$ where the buyer's retention falls below the cutoff, is $-c\beta\Delta^v$, and the corner still dominates: $S(m_b, m) - \sup_{p \leq m_b - M} S = \beta(b - ca)M/(1 + \mu) - c\beta\Delta^v \geq 0$ by (B.37), as in region (b). At pay-all,

$$\Delta_s = v(g(m + m_b)) - v(g(m)) = \frac{b m_b}{1 + \mu} \quad (\text{both points on the } b\text{-branch}),$$

$\Delta_b = \Delta_b(m_b)$ as in (B.38), and

$$S(m_b, m) = \frac{\beta[(1 - c)b m_b + c(b - a)M]}{1 + \mu} - c\beta\Delta^v \geq \frac{(1 - c)\beta b(m_b - M)}{1 + \mu} \geq 0,$$

strictly positive for $m_b > M$, by the upper bound in (B.37).

Step 3: Role values and the hinge function.

Set $\Delta^v = 0$. Equation (B.31) then requires $L(\mu)b = d$; when $L(\mu) > 0$, it gives $b = d/L(\mu) > 0$ and $a = \chi b \in (0, b)$. With the distribution in (B.22)–(B.24) and the bargaining outcome of Step 2, define the two exact one-period role values

$$v_S(m) = -d + \alpha(1 - H)(1 - \theta)[S(\bar{m}_b, m)]^+ + \beta v(g(m)), \quad (\text{B.41})$$

$$v_B(m) = \alpha H \sum_{j \geq 1} w_j \left[\frac{\theta}{c} S(m, \ell_j) \right]^+ + \beta v(g(m)). \quad (\text{B.42})$$

At the continuous member, (B.37) holds and Step 2 shows that the surplus is positive in every relevant match, except for the zero-balance buyer, whose match term is zero. Thus the positive parts may be suppressed in the calculations below. Lemma 4's coefficient equations and the constant-term equations (B.32)–(B.33) give $v_S = v$ on $[0, M]$ and $v_B = v$ on $[M, \infty)$.

For $m \in [0, M]$, define the hinge function

$$X(m) \equiv \sum_{j \geq 1} w_j (\ell_j + m - M)^+ = \sum_{j \geq 1} w_j (m - (1 + \mu)^{-j} M)^+. \quad (\text{B.43})$$

It is continuous, convex, and piecewise linear, with

$$X(0) = 0, \quad X(M) = \bar{m}_s, \quad X'(m) = \sum_{j: (1+\mu)^{-j} M < m} w_j \in [0, 1]$$

away from its kinks. Each summand is a hinge bounded by m , so the series converges uniformly; its active slopes give the displayed derivative. Moreover X is strictly increasing on $(0, M]$: for $0 < m < m' \leq M$, some j satisfies $(1 + \mu)^{-j}M < m$, and that term raises the sum by $w_j(m' - m) > 0$.

We now establish the off-role parts of

$$\begin{aligned} v_S(m) &= \begin{cases} v(m), & m \leq M, \\ v(M) + \frac{\beta b}{1 + \mu}(m - M), & m \geq M, \end{cases} \\ v_B(m) &= \begin{cases} v(m) - A[\bar{m}_s - X(m)], & m \leq M, \\ v(m), & m \geq M, \end{cases} \end{aligned} \tag{B.44}$$

where

$$A \equiv \frac{(1 - \beta + \mu)(b - a)}{(1 - c)(1 + \mu)} > 0.$$

The same calculation will give the anchor, equation (44) of the main text at $\Delta^v = 0$:

$$(1 - \beta)v_0 = \frac{\beta a \mu M}{1 + \mu} + A\bar{m}_s > 0. \tag{B.45}$$

Seller role above M . Consider an off-role seller with $m \geq M$. By the bargaining calculation in Step 2 the meeting trades at pay-all, $p = \bar{m}_b$. We read the value of the index from (B.35). Since $m \geq M$ implies $g(m) \geq M$ and $g(m + \bar{m}_b) \geq M$, both of the seller's drifted positions lie on the b -branch, so

$$\Delta_s = v(g(m + \bar{m}_b)) - v(g(m)) = b[g(m + \bar{m}_b) - g(m)] = \frac{b\bar{m}_b}{1 + \mu},$$

while $\Delta_b = [aM + b(\bar{m}_b - M)]/(1 + \mu)$ from (B.38). Hence, by (B.35),

$$S(\bar{m}_b, m) = \beta\Delta_s - c\beta\Delta_b = \frac{\beta[b\bar{m}_b - c(aM + b(\bar{m}_b - M))]}{1 + \mu} = \frac{\beta[(1 - c)b\bar{m}_b + c(b - a)M]}{1 + \mu},$$

which is *independent of m* : the seller's own balance cancels in the difference Δ_s . Moreover

$S(\bar{m}_b, \cdot)$ is continuous at M : from below,

$$\begin{aligned} S(\bar{m}_b, m) &= \beta \frac{a(M - m) + b(m + \bar{m}_b - M) - c[aM + b(\bar{m}_b - M)]}{1 + \mu} \\ &\xrightarrow{m \uparrow M} \beta \frac{b\bar{m}_b - c[aM + b(\bar{m}_b - M)]}{1 + \mu}, \end{aligned}$$

which equals the off-role value just computed. Hence, subtracting (B.41) at m and at M and using $v_S(M) = v(M)$ (Lemma 4's coefficient equations and Step 1's constant-term equations give $v_S = v$ on $[0, M]$),

$$v_S(m) = v_S(M) + \beta[v(g(m)) - v(g(M))] = v(M) + \frac{\beta b}{1 + \mu}(m - M), \quad m \geq M,$$

since $g(M) = M$ and $v(g(m)) - v(M) = b(m - M)/(1 + \mu)$ on the b -branch. *Sandwich:* $\beta b/(1 + \mu) < b$ from $\beta < 1 + \mu$; and

$$\begin{aligned} a < \frac{\beta b}{1 + \mu} &\iff \chi(1 + \mu) < \beta \\ &\iff s(1 + \mu) < \beta(1 - \beta + \mu + s) \\ &\iff s(1 + \mu - \beta) < \beta(1 - \beta + \mu). \end{aligned}$$

and since $1 + \mu - \beta = 1 - \beta + \mu$, this is $s < \beta$, always true as $s = \alpha\beta(1 - \theta)(1 - H) < \beta$.

Buyer role below M . For $m \in (0, M)$, summing (B.40) over the partner grid with weights w_j gives, with X from (B.43),

$$v_B(m) = \alpha H \frac{\theta \beta}{c} \frac{(1 - c)am + (b - a)X(m)}{1 + \mu} + \beta \left(v_0 + \frac{a(m + \mu M)}{1 + \mu} \right).$$

By (B.28) the coefficient on m (holding the X -term aside) is

$$\frac{(1 - \beta + \mu)a + \beta a}{1 + \mu} = a,$$

so, since $\alpha H \theta \beta (b - a)/[c(1 + \mu)] = A$ by (B.28),

$$v(m) - v_B(m) = \underbrace{(1 - \beta)v_0 - \frac{\beta a \mu M}{1 + \mu}}_{\equiv C} - A X(m), \quad m \in (0, M). \quad (\text{B.46})$$

It remains to pin the constant C . The key is that v_B is *one* function on $[0, \infty)$, the single definition (B.42), and that its two formulas, the off-role one used above for $m < M$ and the on-path one at $m = M$ agree at the cutoff. Indeed, letting $m \uparrow M$ in (B.40), every hinge activates:

$$(\ell_j + m - M)^+ \longrightarrow \ell_j \quad \text{for each } j, \quad \text{so} \quad \Delta_s \longrightarrow \frac{aM + (b-a)\ell_j}{1+\mu},$$

which is exactly the on-path increment in (B.38) at $m_b = M$, $m_s = \ell_j$ (there $\Delta_s = \Delta_b(M) + (b-a)\ell_j/(1+\mu)$ with $\Delta_b(M) = aM/(1+\mu)$). Hence $S(m, \ell_j) \rightarrow S(M, \ell_j)$ for every j ; moreover $X(m) \rightarrow X(M) = \bar{m}_s$ by the hinge-function calculation above, the interchange of limit and sum being justified by dominated convergence (each term is bounded by $\ell_j \leq M$ and $\sum_j w_j = 1$), and $\beta v(g(m)) \rightarrow \beta v(M)$ because v is continuous; *here the continuity of the member does the work*. Therefore v_B is continuous at M , with

$$v_B(M^-) = v_B(M) = \alpha H \frac{\theta}{c} \sum_j w_j S(M, \ell_j) + \beta v(M). \quad (\text{B.47})$$

Now evaluate the right-hand side. Using $S(M, \ell_j) = (1-c)\beta aM/(1+\mu) + \beta(b-a)\ell_j/(1+\mu)$, summing ($\sum_j w_j \ell_j = \bar{m}_s$), and applying (B.28) twice, as $\alpha H \theta \beta (1-c)/c = 1 - \beta + \mu$ on the first term and as $\alpha H \theta \beta /c = (1 - \beta + \mu)/(1-c)$ on the second,

$$v_B(M) = (1 - \beta + \mu) \frac{aM + (b-a)\bar{m}_s/(1-c)}{1+\mu} + \beta(v_0 + aM).$$

Setting $v_B(M) = v(M) = v_0 + aM$ and moving $\beta(v_0 + aM)$ to the left gives precisely

$$(1 - \beta)(v_0 + aM) = (1 - \beta + \mu) \frac{aM + (b-a)\bar{m}_s/(1-c)}{1+\mu},$$

which is (B.32), the constant-term equation established in Step 1. So, *given* (B.32), we have $v_B(M^-) = v(M)$. Since also $v(M^-) = v(M)$, letting $m \uparrow M$ in (B.46) gives $0 = C - A\bar{m}_s$, i.e. $C = A\bar{m}_s$, which is (B.44) for v_B below M . Evaluating the definition at $m = 0$, where no trade occurs by Step 2, and $v_B(0) = \beta v(g(0)) = \beta[v_0 + a\mu M/(1+\mu)]$, the identity $v(0) - v_B(0) = C = A\bar{m}_s$ becomes

$$(1 - \beta)v_0 - \frac{\beta a \mu M}{1 + \mu} = A\bar{m}_s,$$

which is the anchor (B.45). (Equivalently, (B.45) is (B.32) rearranged: subtracting $(1-\beta)aM$ from both sides of (B.32) gives $(1-\beta)v_0 = aM[(1-\beta+\mu) - (1-\beta)(1+\mu)]/(1+\mu) + A\bar{m}_s = \beta\mu aM/(1+\mu) + A\bar{m}_s$.) Both terms are positive for $\mu > 0$, so $v_0 > 0$.

Slope bounds. From (B.44) and the hinge-function calculation above, v_B below M has slopes $a + AX'(m)$ with $X' \in [0, 1]$ nondecreasing: convex piecewise linear, rising from a (near 0) toward $a + A$ (at M^-). Finally

$$a + A < b \iff A < b - a \iff \frac{1 - \beta + \mu}{(1 - c)(1 + \mu)} < 1 \iff \beta > c(1 + \mu),$$

which follows from $H < 1$: $c(1 - \beta + \mu) < \alpha\beta\theta(1 - c) < \beta(1 - c)$ gives $c + c\mu < \beta$.

Step 4: Excluding nonzero jumps.

Write

$$s \equiv \alpha\beta(1 - \theta)(1 - H), \quad C \equiv 1 - \beta + \alpha\beta\theta H > 0.$$

We treat the two signs separately because the bargaining calculation of Step 2 applies directly to an upward jump, whereas a downward jump need not satisfy (B.37).

A downward jump. At $m = M$, let the seller contemplate the pay-all outcome against a buyer with balance m_b . The optimized match surplus, with the option not to trade, is at least the surplus index evaluated at pay-all. That pay-all index is affine in m_b , so its expectation equals its value at $\bar{m}_b$. Moreover,

$$S_{\text{all}}(\bar{m}_b, M) = S_{\text{all}}(\bar{m}_b, 0) + \frac{\beta(b - a)M}{1 + \mu} - \beta\Delta^v.$$

Consequently the actual seller-role value satisfies

$$v_S(M) \geq -d + \alpha(1 - H)(1 - \theta)S_{\text{all}}(\bar{m}_b, M) + \beta v(M).$$

Using (B.30) and $\alpha(1 - H)(1 - \theta)\beta(b - a)/(1 + \mu) = a(1 - \beta + \mu)/(1 + \mu)$ in this lower bound gives

$$v(M) - v_S(M) \leq (1 - \beta + s)\Delta^v. \tag{B.48}$$

Thus $\Delta^v < 0$ implies $v_S(M) > v(M)$, contradicting optimality of the assigned buyer role at M . This argument does not assume that pay-all is the bargaining outcome: optimization and the positive part can only raise the seller-role value above the displayed lower bound.

An upward jump. Suppose $\Delta^v > 0$ in an exact pay-all equilibrium. The retention comparison in region (b) of Step 2 makes

$$\Delta^v \leq \frac{(b - ca)M}{c(1 + \mu)}$$

necessary. Together with $\Delta^v > 0$, this is (B.37); hence all surpluses used below are positive and the positive parts do not bind. The coefficient and constant-term equations imply $v_S = v$ on $[0, M)$ and $v_B = v$ on $[M, \infty)$.

For $m \in [M/(1 + \mu), M)$, every seller-grid partner crosses the cutoff because $\ell_j + m \geq \ell_1 + M/(1 + \mu) = M$. The one-sided jump in the off-role buyer value is

$$v_B(M) - v_B(M^-) = \beta(1 - \alpha\theta H)\Delta^v :$$

the continuation value rises by $\beta\Delta^v$, while each bargained surplus share falls by $\theta\beta\Delta^v$. Since $v(M) - v_B(M) = 0$ and $v(M) - v(M^-) = \Delta^v$, it follows that

$$[v - v_B](M^-) = -C\Delta^v.$$

On the same interval, differentiation and (B.28) give

$$\frac{d}{dm}[v(m) - v_B(m)] = a - \frac{\alpha H \theta \beta}{c} \frac{b - ca}{1 + \mu} - \frac{\beta a}{1 + \mu} = -A.$$

Because $v_S = v$ there, integration back from M^- yields

$$v_S(m) - v_B(m) = -C\Delta^v + A(M - m), \quad m \in [M/(1 + \mu), M). \quad (\text{B.49})$$

Choose n large enough that $\ell_n \geq M/(1 + \mu)$ and $M - \ell_n < C\Delta^v/A$. Such n exists because $\ell_n \uparrow M$. Then (B.49) implies $v_S(\ell_n) < v_B(\ell_n)$ at a seller-grid point with positive stationary mass, contradicting the assigned seller role. Hence an exact equilibrium cannot have either sign of jump, and $\Delta^v = 0$.

Step 5: Strict crossing.

At the continuous member characterized by (B.44),

$$\begin{aligned} v_S(m) - v_B(m) &= A[\bar{m}_s - X(m)] > 0 && \text{on } [0, M), \\ v_B(m) - v_S(m) &= \frac{(1 - \beta + \mu)b}{1 + \mu} (m - M) > 0 && \text{on } (M, \infty), \end{aligned} \quad (\text{B.50})$$

with equality exactly at $m = M$: the two role values cross once, at the cutoff. On $[0, M)$, subtracting the two lines of (B.44) gives $v_S(m) - v_B(m) = v(m) - [v(m) - A(\bar{m}_s - X(m))] = A[\bar{m}_s - X(m)]$, which is strictly positive because X is strictly increasing with $X(M) = \bar{m}_s$, so $X(m) < \bar{m}_s$ for $m < M$. On (M, ∞) ,

$$\begin{aligned} v_B(m) - v_S(m) &= [v(M) + b(m - M)] - \left[v(M) + \frac{\beta b}{1 + \mu} (m - M) \right] \\ &= \left(1 - \frac{\beta}{1 + \mu} \right) b(m - M) = \frac{(1 + \mu - \beta)b(m - M)}{1 + \mu}, \end{aligned}$$

and $1 + \mu - \beta = 1 - \beta + \mu > 0$. At $m = M$ both expressions vanish.

Step 6: Existence, stationarity, and uniqueness. Step 4 has shown that every exact equilibrium has $\Delta^v = 0$. Equation (B.31) therefore becomes $L(\mu)b = d$. If $L(\mu) = 0$, this equation is impossible because $d > 0$; if $L(\mu) < 0$, it requires $b < 0$. Hence no admissible candidate exists when $L(\mu) \leq 0$.

Existence at the continuous member. We verify the requirements in turn. The profile is $\tilde{m} = M$, $b = b^* = d/L(\mu)$, $a = a^* = \chi b^*$, and $(1 - \beta)v_0 = \beta a^* \mu M / (1 + \mu) + A\bar{m}_s$, from (B.45).

(1) *Role optimality and the Bellman equation.* Equations (B.44) and (B.50) give

$$\max\{v_B(m), v_S(m)\} = v(m) \quad \text{for all } m \geq 0.$$

The maximum is attained by S on $[0, M]$ and by B on $[M, \infty)$, strictly except at $m = M$. Thus the Bellman equation in Definition 1 holds and the assigned roles attain the maximum, with indifference only at $m = M$. In particular, the one-shot deviation values are exact and

strictly below v :

$$\begin{aligned} v_B(m) &= v(m) - A[\bar{m}_s - X(m)] && \text{for } m < M, \\ v_S(m) &= v(m) - \frac{(1 - \beta + \mu)b(m - M)}{1 + \mu} && \text{for } m > M. \end{aligned}$$

(2) *Bargaining.* All producer–consumer meetings, on and off the assigned roles, trade at pay-all with $x > 0$ and positive surpluses by Step 2 and (B.38); the only degenerate case is a zero-balance deviating buyer, who does not trade, consistently with $v_B(0) = \beta v(g(0))$ used in (B.45). B – B and S – S meetings contain no producer–consumer pair.

(3) *Monotonicity and the anchor.* $b = d/L(\mu) > 0$ and $a = \chi b \in (0, b)$, so v is strictly increasing; $v_0 > 0$ by (B.45), so no participation issue arises at zero balances.

(4) *Stationarity.* Lemma 3 proves that, when the cutoff is M and trade is pay-all, the seller masses (B.23), the buyer fixed point (33), and their conditional means (34) form the unique stationary distribution under clause 2 of Definition 3. These are exactly the cutoff and trading rule just verified.

(5) *Bellman equations.* (B.32) and (B.33) are the buyer Bellman equation at M and the seller Bellman equation at 0; the remaining on-path Bellman equations hold by Lemma 4’s coefficient calculation and Step 1’s constant-term equations.

Finally, as $\mu \rightarrow 0$, (34) gives $\bar{m}_s \rightarrow 0$ and $\bar{m}_b \rightarrow M/(1 - H_0)$, while $H \rightarrow H_0$, $\chi \rightarrow \chi_0$, and $Z(\mu) \rightarrow Z$. Substitution in (42) yields

$$L(\mu) \longrightarrow ZM \left[\frac{H_0}{1 - H_0} + \chi_0 \right] > 0.$$

Continuity therefore implies $L(\mu) > 0$ for all sufficiently small positive μ .

Uniqueness. The equilibrium is unique in the class of two-branch linear pay-all equilibria, which admits an upward jump at the cutoff; continuity is proven, not assumed. Step 1 of the proof of Lemma 3 pins the cutoff at M ; (B.49) and (B.48) eliminate the jump; (B.31) and (B.45) determine the slopes and anchor; and the proof of Lemma 3 pins the distribution on both sides of the cutoff. All the micro-indeterminacies of the no-policy case, the cutoff, the jump, and the distribution, with the slopes they induce, are resolved by the policy. $\square$

B.3 Proofs for Aggregate Equations and Welfare

This appendix proves Proposition 4; the proof of Proposition 5 is in the main text, and the computer based proof of Proposition 6 is in Appendix C.1.

B.3.1 The aggregate value equations: proof of Proposition 4

Proof. Maintain the stationary equilibrium characterized in Proposition 3. Hence $\mu > 0$, $H^*(L) \in (0, 1)$, and $L(\mu) > 0$, and the hypotheses of Lemmas 3 and 4 hold. In particular, (34), (40), and (41) apply.

Write the buyer branch of the value function in cutoff-centered form as $v(m) = v(0) + aM + \Delta^v + b(m - M)$ for $m \geq M$. The derivation uses only linearity on each branch, so it applies to every candidate profile. Throughout the proof, write $H \equiv H^*(L)$, $a \equiv a^*$, and $g(m) \equiv (m + \mu M)/(1 + \mu)$, and recall $\tilde{\theta} = \theta/K$ and $1 - \tilde{\theta} = c(1 - \theta)/K$ with $K = c + \theta - c\theta$.

Step 1: Inflation-drift averages Under the value function (35), for a buyer $m \geq M$ the continuation balance $g(m)$ stays in $[M, \infty)$, so

$$v(g(m)) = v(m) - \frac{b\mu(m - M)}{1 + \mu}.$$

Averaging over the buyer distribution,

$$\frac{1}{1 - H} \int_M^\infty v(g(m)) dH^*(m) = \bar{v}_b^* - \Gamma_b. \quad (\text{B.51})$$

For a seller $m < M$ the continuation balance stays in $[0, M)$, so $v(g(m)) = v(m) + a\mu(M - m)/(1 + \mu)$ and

$$\frac{1}{H} \int_0^M v(g(m)) dH^*(m) = \bar{v}_s^* + \Gamma_s. \quad (\text{B.52})$$

Finally, $g(0) = \mu M/(1 + \mu) < M$ gives

$$v(g(0)) = v(0) + \frac{a\mu M}{1 + \mu}, \quad (\text{B.53})$$

Linearity of v on $[0, M)$ also gives

$$v(0) = \bar{v}_s^* - a \bar{m}_s. \quad (\text{B.54})$$

Step 2: The two average value equations Average the buyer Bellman equation (38) over the buyer distribution. The match term integrates to

$$\frac{\alpha \tilde{\theta}}{1 - H} \int_M^\infty \int_0^M J(m_b, m_s) dH^*(m_s) dH^*(m_b) = \alpha H \tilde{\theta} \bar{J}^*,$$

and the continuation term is $\beta(\bar{v}_b^* - \Gamma_b)$ by (B.51); rearranging yields (48). Averaging the seller Bellman equation (39) over the seller distribution, the match term integrates to $\alpha(1 - H)(1 - \tilde{\theta}) \bar{J}^*$ and the continuation term is $\beta(\bar{v}_s^* + \Gamma_s)$ by (B.52); rearranging yields (49).

Step 3: The average surplus equation By the bargaining characterization of Appendix B.2, applied with the inflation-adjusted continuation values of this section (the surpluses in (38)–(39) are defined relative to the post-drop balances $g(\cdot)$), on-path production under pay-all ($p = m_b$) solves

$$c x(m_b, m_s) = c(1 - \theta) A(m_b) + \theta B(m_b, m_s),$$

where

$$A(m_b) \equiv \beta[v(g(m_b)) - v(g(0))], \quad B(m_b, m_s) \equiv \beta[v(g(m_b + m_s)) - v(g(m_s))]$$

are the buyer's forgone continuation gain and the seller's continuation gain from the payment. Under (35),

$$B(m_b, m_s) - A(m_b) = \frac{\beta(b - a)}{1 + \mu} m_s,$$

which is exactly the wedge in (37). Hence

$$J = (1 - c)x + (B - A) = [(1 - c)(1 - \theta) - 1]A + \left[\frac{(1 - c)\theta}{c} + 1\right]B = \frac{K}{c}(B - cA),$$

using $(1 - c)(1 - \theta) - 1 = -K$ and $(1 - c)\theta/c + 1 = K/c$. Averaging over matches, Step 1, (B.53), and (B.54) give

$$\begin{aligned} E[A] &= \beta \left[\bar{v}_b^* - \Gamma_b - v(0) - \frac{a\mu M}{1 + \mu} \right] = \beta \left[\bar{v}_b^* - \bar{v}_s^* - \Gamma_b + a \bar{m}_s - \frac{a\mu M}{1 + \mu} \right], \\ E[B] &= E[A] + \frac{\beta(b - a) \bar{m}_s}{1 + \mu}. \end{aligned}$$

Therefore

$$\bar{J}^* = \frac{K}{c} E[B - cA] = \frac{K}{c} \beta \left[(1 - c)(\bar{v}_b^* - \bar{v}_s^* - \Gamma_b) + (1 - c) \left(a \bar{m}_s - \frac{a\mu M}{1 + \mu} \right) + \frac{(b - a) \bar{m}_s}{1 + \mu} \right].$$

The last two terms in the bracket combine to $\Gamma_J - (1 - c)\Gamma_s$: indeed,

$$\Gamma_J - (1 - c)\Gamma_s = \frac{(b - ca) \bar{m}_s}{1 + \mu} - \frac{(1 - c) a \mu (M - \bar{m}_s)}{1 + \mu} = \frac{(b - a) \bar{m}_s}{1 + \mu} + (1 - c) a \bar{m}_s - \frac{(1 - c) a \mu M}{1 + \mu},$$

where the second equality collects the $\bar{m}_s$ terms using $(b - ca) + (1 - c)a\mu = (b - a) + (1 - c)a(1 + \mu)$. This proves (50).

Step 4: Closed form for average surplus Subtracting (49) from (48),

$$(1 - \beta)(\bar{v}_b^* - \bar{v}_s^*) = d + \frac{\alpha}{K} [\theta H - c(1 - \theta)(1 - H)] \bar{J}^* - \beta(\Gamma_b + \Gamma_s),$$

where we used $\tilde{\theta} = \theta/K$ and $1 - \tilde{\theta} = c(1 - \theta)/K$. Solving (50) for the value gap,

$$\bar{v}_b^* - \bar{v}_s^* = \frac{c \bar{J}^*}{K\beta(1 - c)} - \frac{\Gamma_J}{1 - c} + \Gamma_b + \Gamma_s,$$

and substituting into the previous display,

$$\frac{c \bar{J}^*}{K\beta(1 - c)} \left\{ (1 - \beta) - \alpha\beta(1 - c) \left[\frac{\theta H}{c} - (1 - \theta)(1 - H) \right] \right\} = d + \frac{(1 - \beta) \Gamma_J}{1 - c} - (\Gamma_b + \Gamma_s),$$

where the braces equal $\mathcal{D}$ in (47). When $\mathcal{D} \neq 0$, multiplying through by $K\beta(1 - c)/(c\mathcal{D})$ yields (51); the two value averages then follow from (48) and (49).

Step 5: Welfare accounting identities. Finally, we record the derivation of the welfare identity

(53). Substituting (48)–(49) into $W(\mu) = H \bar{v}_s^* + (1 - H) \bar{v}_b^*$ and using $\tilde{\theta} + (1 - \tilde{\theta}) = 1$ gives

$$(1 - \beta)W(\mu) = \alpha H(1 - H) \bar{J}^* - H d + \beta [H \Gamma_s - (1 - H) \Gamma_b],$$

which is (53). Conversely, averaging (37) gives $\bar{J}^* = (1 - c)\bar{x}^* + \beta(b - a)\bar{m}_s/(1 + \mu)$, and (54) shows that the drift terms cancel against the average wedge, recovering (52). □

B.4 Proofs for Section 6

This appendix proves the on-path construction and its aggregate implications, then verifies Proposition 9 in one argument. Throughout, write $s \equiv \alpha\beta(1 - \theta)(1 - H_0)$, so that $\chi = s/(1 - \beta + s)$ and $Z = s(1 - c)$.

B.4.1 Proof of Proposition 7

Proof. Maintain the standing pre-injection conditions of Section 6 and conditions (R') and (T). Hence the baseline identities (68)–(71) apply to the initial member used below.

Time-invariant slopes. Matching the coefficient of the agent's own balance in the on-path buyer equation and using (68) gives

$$b_t = \left[\beta + \frac{\alpha H_0 \theta \beta (1 - c)}{c} \right] b_{t+1} = b_{t+1}. \quad (\text{B.55})$$

Thus b_t is constant, but neither boundedness nor the buyer recursion determines its level. We maintain the selection $b_t = b^*$, equal to the pre-announcement slope. Given this selection, coefficient matching on the seller branch gives

$$a_t = s b_{t+1} + (\beta - s) a_{t+1}. \quad (\text{B.56})$$

Because $0 < \beta - s < 1$, every deviation from the fixed point $s b^*/(1 - \beta + s) = \chi b^* = a^*$ grows without bound when the recursion is run forward. The unique bounded solution is therefore $a_t = a^*$.

On-path support and the injection pool. At date 1, every initial seller, of mass H_0 , holds

$\tau < \tilde{m}$, and every initial buyer holds at least $\tilde{m} + \tau$. In a candidate pay-all trade, the buyer becomes a zero-balance seller and the seller receives the buyer's balance and enters the buyer region. A trade therefore replaces one seller with another and leaves the seller mass at H_0 . A seller holding τ leaves that pool only by meeting a buyer, with probability $\alpha(1 - H_0)$, and the pool is never replenished. This proves (74). The conditional seller mean is $h_t\tau$, and money-market clearing,

$$(1 - H_0)\overline{m}_{b,t} + H_0h_t\tau = (1 + \mu)M, \quad (\text{B.57})$$

gives (75).

Bellman reduction. Average (77) over the seller mix, whose mean is $h_t\tau$. The buyer equation becomes

$$\begin{aligned} v_t(m_b) = & \left[\frac{\alpha H_0 K \beta}{c} + (1 - \alpha H_0) \beta \right] \Delta_{b,t} + \beta v_{0,t+1} \\ & + \frac{\alpha H_0 \theta \beta}{c} (b^* - a^*) h_t \tau. \end{aligned} \quad (\text{B.58})$$

The coefficient on $\Delta_{b,t}$ is one because $K - c = \theta(1 - c)$ and (68) holds. Evaluating the left side with (72) and cancelling the slope terms gives (78).

For a seller at $m_s < \tilde{m}$, the candidate role value is

$$\begin{aligned} v_{S,t}(m_s) = & -d + \beta(v_{0,t+1} + a^* m_s) \\ & + \alpha(1 - H_0)(1 - \theta)\beta \left[(1 - c)\overline{\Delta}_{b,t} + (b^* - a^*)m_s \right], \end{aligned} \quad (\text{B.59})$$

where $\overline{\Delta}_{b,t}$ is (76) evaluated at the buyer mean. The coefficient on m_s is $\beta a^* + \alpha(b^* - a^*) = a^*$, and the constant term is (79). Together with (74), this proves the three-equation system for both seller types and all on-path buyers.

Steady state and roots. Setting $h = 0$ in (82) gives (83); subtracting (70) gives (84). For the centered system, the characteristic equation factors as

$$0 = (\gamma_3 - \lambda)(1 - \lambda\beta)[1 - \lambda(1 - Z)]. \quad (\text{B.60})$$

This gives the three roots in (87). Only γ_3 is stable. The system has the single predetermined variable $h_1 = 1$ and two free level variables, so the convergent path is unique.

Normalize its stable eigenvector as $(\varphi_\Delta, \varphi_v, 1)^\top$. The first two rows of the eigenvector equation are

$$\begin{aligned} (1 - \gamma_3)\varphi_\Delta + (1 - \beta\gamma_3)\varphi_v &= K_0\tau, \\ -\gamma_3 Z\varphi_\Delta + (1 - \beta\gamma_3)\varphi_v &= -\frac{Zb^*H_0\tau}{1 - H_0}. \end{aligned} \quad (\text{B.61})$$

Subtracting the second equation from the first gives φ_Δ in (89); the second then gives φ_v in (90). With $h_1 = 1$, the stable path is (88). $\square$

B.4.2 Proof of Proposition 8

Proof. Maintain the standing pre-injection conditions and conditions (R') and (T). Proposition 7 therefore supplies the candidate path (88) and the on-path distribution used below.

Averaging (77) over buyers and the seller mix gives

$$\bar{x}_t = \frac{K\beta}{c}\bar{\Delta}_{b,t} + h_t \frac{\theta\beta}{c}(b^* - a^*)\tau. \quad (\text{B.62})$$

By (88) and (75),

$$\begin{aligned} \bar{\Delta}_{b,t} &= \frac{d}{Z} + \left[\gamma_3\varphi_\Delta - \frac{b^*H_0\tau}{1 - H_0} \right] h_t \\ &= \frac{d}{Z} + \frac{(1 - \beta\gamma_3)\varphi_v}{Z} h_t. \end{aligned} \quad (\text{B.63})$$

Substitution proves (91)–(93).

Pay-all makes average nominal expenditure per trade equal to $\bar{m}_{b,t}$, which gives (94). Put $z = \gamma_3^{t-1}$ and write the numerator and denominator of p_t as $N_0 + N_1z$ and $X_0 + X_1z$, where

$$N_0 = \frac{(1 + \mu)M}{1 - H_0}, \quad N_1 = -\frac{H_0\tau}{1 - H_0}, \quad X_0 = \bar{x}_\infty, \quad X_1 = \hat{x}_1. \quad (\text{B.64})$$

The derivative with respect to z has the constant sign of $N_1X_0 - N_0X_1$. Using $\tau = \mu M$, this expression is negative exactly when $\hat{x}_1 > \hat{x}_L$. Comparing p_1 with p_{pre} shows that $p_1 > p_{\text{pre}}$ exactly when $\hat{x}_1 < \hat{x}_U$. These two comparisons give the three price cases.

The seller mass is constant, so the sunk-cost flow cancels when welfare is measured

relative to the terminal allocation. Hence

$$\begin{aligned}\Delta W &= (1 - c) \sum_{t \geq 1} \beta^{t-1} (Y_t - Y_\infty) \\ &= (1 - c) \alpha H_0 (1 - H_0) \hat{x}_1 \sum_{t \geq 1} (\beta \gamma_3)^{t-1},\end{aligned}\tag{B.65}$$

which is (96). □

B.4.3 Proof of Proposition 9

Proof. Maintain the standing pre-injection conditions and conditions (R'), (T), and (C). Proposition 7 therefore supplies the continuation profiles and on-path support used below. Parts (ii) and (iii) additionally impose condition (I), exactly where stated in the proposition.

Pay-all. Apply the baseline bargaining characterization (B.13)–(B.14) at date t to the continuation profile v_{t+1} , with $(a, b, \Delta^v) = (a^*, b^*, \Delta_{t+1}^v)$. Equations (69) and (71) give $0 < a^* < b^*$. Under (R'), on-path sellers hold 0 or $\tau < \tilde{m}$, while on-path buyers hold at least $\tilde{m} + \tau$. Condition (T) and Proposition 7 give $\Delta_{t+1}^v \geq 0$ and show that the largest continuation jump is Δ_2^v . Condition (C) therefore supplies the upper bound in (B.13) at every date. The characterization implies that pay-all is optimal and surplus is strictly positive in every on-path meeting. This proves part (i).

Role gaps and condition (I). For any m with $m + \tau < \tilde{m}$, a deviation to B leaves both the agent and either seller partner below the cutoff. For every feasible payment, the surplus index in (B.12) therefore has slope $(1 - c)\beta a^* > 0$, so pay-all is optimal. The deviating role value is

$$v_{B,t}(m) = \beta v_{0,t+1} + a^* m,\tag{B.66}$$

where (68) makes the coefficient on m equal to a^* . Since the prescribed seller value is $v_{S,t}(m) = v_{0,t} + a^* m$, (78) and (88) give (98). The same calculation shows that this role-gap formula holds for every m with $m + \tau < \tilde{m}$.

For $m \geq \tilde{m}$, pay-all follows from the off-role-seller argument in Step 3 of Appendix B.1.1, using the bound (B.13). Subtracting the deviating seller value from the high branch and

using (79) gives

$$v_{B,t}(m) - v_{S,t}(m) = (1 - \beta)b^*(m - \tilde{m}) + (1 - \beta + s)\Delta_{t+1}^v + (\Delta_t^v - \Delta_{t+1}^v) > 0. \quad (\text{B.67})$$

The jump makes the buyer role strict even at the cutoff.

Substituting (89) into the coefficient of h_t in (98) and using (90) gives two expressions with the same sign:

$$K_0\tau - (1 - \gamma_3)\varphi_\Delta = \frac{Z\gamma_3\tau}{1 - \gamma_3 + Z\gamma_3} \left[K_0 - \frac{(1 - \gamma_3)b^*H_0}{\gamma_3(1 - H_0)} \right], \quad (\text{B.68})$$

$$\varphi_v = \frac{Z\tau}{(1 - \beta\gamma_3)(1 - \gamma_3 + Z\gamma_3)} \left[\gamma_3 K_0 - \frac{(1 - \gamma_3)b^*H_0}{1 - H_0} \right]. \quad (\text{B.69})$$

Using $K_0 = (1 - \beta)(1 - \chi)b^*/(1 - c)$, $1 - \gamma_3 = \alpha(1 - H_0)$, and $\alpha H_0 = c(1 - \beta)/[\beta\theta(1 - c)]$, the common sign reduces to

$$\text{sign} \left[\frac{(1 - \beta)b^*}{1 - c} \left((1 - \chi)\gamma_3 - \frac{c}{\beta\theta} \right) \right] = \text{sign}[\beta\theta\gamma_3(1 - \chi) - c]. \quad (\text{B.70})$$

This proves (99). Under (I), on-path seller gaps and buyer gaps are strictly positive, and $v_{0,t} = \varphi_v h_t > 0$, so every on-path value is positive. Under the reverse inequality, a zero-balance agent can choose B forever, never trade, and obtain zero instead of the negative value $v_t(0)$. This proves part (ii).

Output and welfare. Under (I), $\varphi_v > 0$ by (99). Both terms in (92) are then strictly positive, so $\hat{x}_1 > 0 > \hat{x}_L$. Equation (96) gives $\Delta W > 0$, and price case (c) is excluded. This proves part (iii).

The off-path window. For $m \in [\tilde{m} - \tau, \tilde{m})$, pay-all against a zero-balance partner follows from the slope $(1 - c)\beta a^* > 0$. Against a τ -partner, the index has that slope before the crossing, jumps by $\beta\Delta_{t+1}^v \geq 0$ there, and has slope $\beta(b^* - ca^*) > 0$ afterward. Hence pay-all is optimal, and the crossing adds the continuation gain

$$G_t(m) \equiv \Delta_{t+1}^v + (b^* - a^*)(m - (\tilde{m} - \tau)). \quad (\text{B.71})$$

A fraction h_t of seller partners holds τ , and Nash bargaining passes the corresponding reval-

uation to the deviating buyer. Therefore

$$\begin{aligned} v_{S,t}(m) - v_{B,t}(m) &= h_t [K_0 \tau - (1 - \gamma_3) \varphi_\Delta] \\ &\quad - h_t \frac{1 - \beta}{1 - c} [\Delta_{t+1}^v + (b^* - a^*)(m - (\tilde{m} - \tau))] . \end{aligned} \quad (\text{B.72})$$

The expression is strictly decreasing in m . Its zero is

$$\underline{m}_t \equiv \tilde{m} - \frac{(1 - \gamma_3) \varphi_\Delta}{K_0} - \frac{\Delta_{t+1}^v}{b^* - a^*}. \quad (\text{B.73})$$

As $m \uparrow \tilde{m}$, the bracketed gap divided by h_t converges to

$$-(1 - \gamma_3) \varphi_\Delta - \frac{1 - \beta}{1 - c} \Delta_{t+1}^v < 0. \quad (\text{B.74})$$

Hence the seller assignment fails on $(\max\{\underline{m}_t, \tilde{m} - \tau\}, \tilde{m})$, a nonempty interval of width at most τ at every finite date. The violation carries the common factor $h_t = \gamma_3^{t-1}$ and vanishes as the injection is absorbed. Under (R'), on-path balances avoid this window, and the preceding role-gap calculations cover every balance outside it. This proves part (iv) and the stated scope of the construction. $\square$

C.1 Computer-Based Proof of Proposition 6

This appendix is separated from the analytical appendices because its proof is computer based. Every claim is reduced to either a finite exact computation or a polynomial-coefficient certificate, and both are carried out by the SymPy program printed in Section C.1.2. The program works in exact rational arithmetic, reconstructs every object from the closed forms displayed in the main text, and verifies the required identities and coefficient signs with `assert` statements; a successful run prints `all checks passed`. The role of the text is only to state what the program computes and why the computations imply the proposition.

Throughout this appendix, maintain (30). For every $\mu \in (0, \bar{\mu})$, the definition of $\bar{\mu}$ gives $H(\mu) \in (0, 1)$ and $L(\mu) > 0$. Proposition 3 therefore supplies the unique equilibrium, and Lemmas 3–4 and Proposition 4 justify the closed-form expressions used by the program. The boundary claims use the one-sided limits of these same expressions as $\mu \uparrow \bar{\mu}$.

C.1.1 What the program certifies

The proof consists of the six items below. Items 1–4 are exact symbolic computations; the endpoint identities, factorization, and Bernstein certificate are checked by **assert** statements in the correspondingly named blocks of the program. Items 5 and 6 are short analytic arguments from the computed quantities. The program assembles $W(\mu)$ from the closed forms (40), (41), (34), and (42), together with average production per match,

$$c\bar{x}^* = \beta \left[K \bar{\Delta}_b + \theta(b^* - a^*) \frac{\bar{m}_s}{1 + \mu} \right], \quad \bar{\Delta}_b = \frac{a^*M + b^*(\bar{m}_b - M)}{1 + \mu}, \quad (\text{C.1})$$

and the welfare definition (52); it uses nothing else.

1. *Endpoints of L (block **endpoints**)*. The program verifies the exact identity (60) for $L(0)$ and

$$L(\mu_H) = -\frac{(1 - \beta + \mu_H)M}{(1 - c)(1 + \mu_H)}. \quad (\text{C.2})$$

Under (30), $H_0 \in (0, 1)$, so every factor of $L(0)$ is strictly positive and $L(\mu_H)$ is strictly negative.

2. *Coordinates and the total derivative (block **coordinates**)*. The program substitutes

$$p = H, \quad q = \frac{\mu}{1 - \beta + \mu}, \quad \text{inverse} \quad \mu = \frac{q(1 - \beta)}{1 - q}, \quad \alpha = \frac{c(1 - \beta)}{(1 - q)p\beta\theta(1 - c)}, \quad (\text{C.3})$$

and computes $dW(\mu)/d\mu$ as the total derivative along $dp/d\mu = p(1 - q)/(1 - \beta)$ and $dq/d\mu = (1 - q)^2/(1 - \beta)$: a change in μ moves the extensive margin p and the drift margin q together.

3. *Factorization (block **factorization**)*. The program factors the resulting rational function completely and verifies

$$\frac{dW(\mu)}{d\mu} = \frac{dc\theta p(1 - q)}{(1 - \beta)^2} \frac{\tilde{P}(p, q, \beta, \theta, c)}{G(p, q, \beta, \theta, c)^2}, \quad (\text{C.4})$$

where $\tilde{P}$ is a polynomial of multidegree $(6, 2, 2, 5, 4)$ in (p, q, β, θ, c) and G is exactly the numerator polynomial of $L(\mu)$ in these coordinates (also verified). The denominator is a perfect square, finite and nonzero wherever the equilibrium exists, and the prefactor is

positive, so on the admissible set

$$\text{sign } \frac{dW(\mu)}{d\mu} = \text{sign } \tilde{P}. \quad (\text{C.5})$$

$\tilde{P}$ is too large to display, and no display is needed: the program reconstructs it from the closed forms listed above, so the certificate below, not the polynomial, is the checkable object.

4. *The Bernstein certificate (block **Bernstein**)*. Every polynomial of multidegree $n = (n_1, \dots, n_5)$ has a unique expansion in the tensor-product Bernstein basis on $[0, 1]^5$,

$$\tilde{P} = \sum_{\mathbf{i}} p_{\mathbf{i}} \prod_{d=1}^5 \binom{n_d}{i_d} x_d^{i_d} (1 - x_d)^{n_d - i_d}, \quad x = (p, q, \beta, \theta, c). \quad (\text{C.6})$$

Each basis polynomial is nonnegative on the closed box and strictly positive at every interior point; hence if every $p_{\mathbf{i}} \geq 0$ and not all vanish, the polynomial is strictly positive on the open box. The program computes all $7 \cdot 3 \cdot 3 \cdot 6 \cdot 5 = 1,890$ Bernstein coefficients of $\tilde{P}$ in exact rational arithmetic and verifies that none is negative and 335 are strictly positive. Therefore $\tilde{P} > 0$ on $(0, 1)^5$, and by (C.5) part (i) of Proposition 6 holds at every admissible μ . (For a positive polynomial the base-degree coefficients need not be nonnegative in general; that no degree elevation or box subdivision is needed here is a feature of $\tilde{P}$, and is itself part of what the program verifies.)

5. *The first zero of L (analytic, from item 1)*. L is continuous on $[0, \mu_H]$, positive at 0 and negative at μ_H , so its zero set in $[0, \mu_H]$ is nonempty and compact; its minimum $\bar{\mu}$ satisfies $0 < \bar{\mu} < \mu_H$, $L > 0$ on $[0, \bar{\mu})$, and $H < 1$ there. This proves (65) and, with item 4, part (i).

6. *Divergence (analytic)*. Substituting the slope $b^* = d/L(\mu)$ into (C.1) and (52) gives $W(\mu) = (d/(1 - \beta))[\Psi(\mu)/L(\mu) - H]$ with

$$\Psi(\mu) = \frac{(1 - H)(1 - \beta + \mu)}{1 + \mu} \left\{ \frac{K}{\theta} [\chi M + (\bar{m}_b - M)] + (1 - \chi)\bar{m}_s \right\}, \quad (\text{C.7})$$

a product and sum of factors that are strictly positive at $\bar{\mu}$, where $H \in (0, 1)$. As $\mu \uparrow \bar{\mu}$, $L(\mu) \rightarrow 0^+$ while Ψ converges to a positive limit, so $b^* = d/L(\mu)$ and $W(\mu)$ diverge to $+\infty$. This proves (66).

C.1.2 The program

The following Python program is the proof. It is self-contained, uses SymPy (version 1.14.0), runs in well under a minute, and must print all checks passed.

```
# Computer-based certificate for Proposition 6:
# (i)  $dW/d\mu > 0$  at every admissible  $\mu$  in  $(0, \mu_{\text{bar}})$ ;
# (ii)  $L(0) > 0$  and  $L(\mu_H) < 0$ , so the first zero  $\mu_{\text{bar}}$  of  $L$  lies in
#  $(0, \mu_H)$ , strictly before  $H = 1$ .
# Exact rational and symbolic arithmetic throughout; no floating point
# and no randomness.
# Run in a fresh interpreter; every assert must pass.
import itertools
from math import comb
from fractions import Fraction
import sympy as sp

al, be, th, c, mu, d, M = sp.symbols(
    'alpha beta theta c mu d M', positive=True)
p, q = sp.symbols('p q', positive=True)

# ---- assembly of  $W(\mu)$  from the displayed closed forms ----
H = c*(1-be+mu)/(al*be*th*(1-c))          # seller mass
s = al*be*(1-th)*(1-H)
chi = s/(1-be+mu+s)                       # slope ratio
Z = al*be*(1-c)*(1-th)*(1-H)              #  $Z(\mu)$ 
ms = mu*M/(mu+al*(1-H))                  # seller mean
mb = (mu+al)*M/(mu+al*(1-H))              # buyer mean
L = chi*M*((1-be)+be*mu/(1+mu)) + Z*((mb-M)+chi*M)/(1+mu) \
    - (1-be+mu)*(chi*M+(1-chi)*ms/(1-c))/(1+mu)      #  $L(\mu)$ 
b = d/L; a = chi*b; K = c+th-c*th
Db = (a*M+b*(mb-M))/(1+mu)               # avg discounted balance
x = (be/c)*(K*Db + th*(b-a)*ms/(1+mu))   # production per match
W = (al*H*(1-H)*(1-c)*x - H*d)/(1-be)    # welfare
```

```

# ---- endpoints of L as exact identities ----
H0s = c*(1-be)/(al*be*th*(1-c))
s0 = al*be*(1-th)*(1-H0s); chi0 = s0/(1-be+s0)
Z0 = al*be*(1-c)*(1-th)*(1-H0s)
assert sp.cancel(L.subs(mu, 0) - Z0*M*(H0s/(1-H0s)+chi0)) == 0
muH = al*be*th*(1-c)/c - (1-be)
assert sp.cancel(L.subs(mu, muH) + (1-be+muH)*M/((1-c)*(1+muH))) == 0

# ---- the (p,q) coordinates and the total derivative ----
mu_s = q*(1-be)/(1-q)
al_s = c*(1-be)/((1-q)*p*be*th*(1-c))
Wpq = sp.cancel(sp.together(W.subs({mu: mu_s, al: al_s})))
dWpq = sp.cancel(sp.together(sp.diff(Wpq,p)*p*(1-q)/(1-be)
                               + sp.diff(Wpq,q)*(1-q)**2/(1-be)))

# ---- factorization: dW/dmu = c th p (1-q) Ptil / ((1-be)^2 G^2) ----
N, D = sp.fraction(dWpq)
g = sp.gcd(N, D)
N = sp.expand(sp.cancel(N/g/d)); D = sp.expand(sp.cancel(D/g))
fl = sp.factor_list(N)
Ptil = [f for f, m_ in fl[1] if sp.degree(f, p) == 6][0]
assert sp.expand(N - c*th*p*(1-q)*Ptil) == 0
G = sp.factor_list(sp.cancel(D/(be-1)**2))[1][0][0]
assert sp.expand(D - (be-1)**2*G**2) == 0
Lpq = sp.cancel(sp.together(L.subs({mu: mu_s, al: al_s})))
F3 = c*(1-th)*(1-p)+th*(1-c)*p
F4 = c*(1-p)+be*th*(1-c)*p*q
assert sp.expand(sp.cancel(Lpq*p*th*(1-be*q)*F3*F4/(M*(1-be))) + G) == 0

# ---- the Bernstein certificate: Ptil > 0 on the open box (0,1)^5 ----
def bern(poly, vs):
    n = tuple(sp.degree(poly, v) for v in vs)
    A = {mo: Fraction(sp.Rational(cf).p, sp.Rational(cf).q)
          for mo, cf in sp.Poly(poly, *vs).terms()}

```

```

neg = pos = 0
for i in itertools.product(*[range(nd+1) for nd in n]):
    sacc = Fraction(0)
    for k, aco in A.items():
        if all(kd <= id_ for kd, id_ in zip(k, i)):
            w = Fraction(1)
            for kd, id_, nd in zip(k, i, n):
                w *= Fraction(comb(id_, kd), comb(nd, kd))
            sacc += w*aco
    neg += sacc < 0; pos += sacc > 0
return n, pos, neg
n_, pos, neg = bern(sp.expand(Ptil), (p,q,be,th,c))
assert n_ == (6,2,2,5,4) and neg == 0 and pos == 335

print("all checks passed")

```